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ON VAN DER CORPUT'S METHOD AND THE ZETA-FUNCTION OF RIEMANN

By E. C. TITCHMARSH

[Received 8 May 1931]

1. THE researches of van der Corput on the divisor problem and other problems in the analytic theory of numbers are well known,* and his methods also give very profound and interesting results concerning the zeta-function of Riemann. His work depends on the construction of inequalities satisfied by sums of the form

$$\sum e^{2\pi i f(n)} \tag{1}$$

where $f(x)$ is a real function. We give here a simplified form of van der Corput's method, which leads to an alternative inequality for sums of the form (1). The result is formula (1) of § 4; it has two terms on the right-hand side instead of the three which occur in van der Corput's formulae. While the analysis is materially shortened, the results to which it leads in the theory of the zeta-function and other problems are the same as those of van der Corput.

In the theory of the 'circle-problem', for example, we prove that, if $R(x)$ is the number of solutions in integers of the inequality $u^2+v^2 \leqslant x$, then

$$R(x) = \pi x + O\left(x^{\frac{27}{82}}\right). \tag{2}$$

This is the result obtained by Nieland.

In the theory of the Riemann zeta-function, it was proved by van der Corput and Koksma that

$$\zeta(\sigma+it) = O\left\{\left(\frac{t^{1/(4Q-2)}}{\log 1/(1-\sigma)} + \frac{1}{\log\log t}\right)\log t\right\} \tag{3}$$

where $Q = 2^{q-1}$, uniformly on the lines $\sigma = 1-(q+1)/(4Q-2)$, for $q = 2, 3, 4, \ldots$. In order to exhibit the process in its simplest form we shall consider fixed values of σ only, and prove that on each of the above lines

$$\zeta(\sigma+it) = O(t^{1/(4Q-2)}\log t). \tag{4}$$

This is the most that is known in this direction, apart from the isolated result, deduced by Walfisz from another theorem of van der Corput,† that

$$\zeta(\tfrac{1}{2}+it) = O\left(t^{\frac{163}{988}}\right).$$

* See the list of papers at the end.

† Van der Corput (**2**), Walfisz (**1**).

For the sake of clearness we give the whole proof, though it is only in the later stages that there is any novelty in the method.

2. We begin by proving some lemmas. In what follows A denotes an absolute constant, not necessarily the same one at each occurrence; and the symbol O implies an absolute constant except in §§ 4–4.1, where the constant may depend on k, q, and σ.

2.1. LEMMA 1.* *Let $f(x)$ be a real differentiable function in the interval (a, b), $f'(x)$ be monotonic in this interval, and $|f'(x)| \leqslant \frac{1}{2}$. Then*

$$\Big| \sum_{a \leqslant n \leqslant b} e^{2\pi i f(n)} - \int_a^b e^{2\pi i f(x)}\, dx \Big| < A. \tag{1}$$

In the proof we may take a and b to be integers; for the left-hand side is changed by $O(1)$ if we replace a and b by the nearest integers.

Let
$$\chi(x) = -\frac{1}{\pi} \sum_{\nu=1}^{\infty} \frac{\sin 2\nu\pi x}{\nu}.$$

This series is boundedly convergent, so that we may multiply by a bounded function and integrate term-by-term. Thus

$$2\pi i \int_a^b \chi(x) e^{2\pi i f(x)} f'(x)\, dx = -\sum_{\nu=1}^{\infty} \frac{2i}{\nu} \int_a^b \sin 2\nu\pi x\, e^{2\pi i f(x)} f'(x)\, dx$$

$$= -\frac{1}{2\pi i} \sum_{\nu=1}^{\infty} \frac{1}{\nu} \int_a^b \left\{ \frac{f'(x)}{f'(x)+\nu}\, d(e^{2\pi i\{f(x)+\nu x\}}) - \frac{f'(x)}{f'(x)-\nu}\, d(e^{2\pi i\{f(x)-\nu x\}}) \right\}. \tag{2}$$

Since each of the functions $f'(x)/\{f'(x) \pm \nu\}$ is monotonic and does not exceed $1/(2\nu-1)$ in absolute value, it follows from the second mean-value theorem that the right-hand side of (2) is

$$\sum_{\nu=1}^{\infty} \frac{1}{\nu} \cdot O\left(\frac{1}{2\nu-1}\right) = O(1).$$

Also $\chi(x) = x - [x] - \frac{1}{2}$ if x is not an integer; hence the left-hand side of (2) is equal to

$$2\pi i \int_a^b (x - [x] - \tfrac{1}{2}) e^{2\pi i f(x)} f'(x)\, dx.$$

Now

$$2\pi i \int_a^b (x - \tfrac{1}{2}) e^{2\pi i f(x)} f'(x)\, dx = (b - \tfrac{1}{2}) e^{2\pi i f(b)} - (a - \tfrac{1}{2}) e^{2\pi i f(a)} - \int_a^b e^{2\pi i f(x)}\, dx,$$

* Van der Corput (1), 58.

and $$2\pi i\int_a^b [x]e^{2\pi i f(x)}f'(x)\,dx=\sum_{n=a}^{b-1}2\pi in\int_n^{n+1}e^{2\pi i f(x)}f'(x)\,dx$$

$$=\sum_{n=a}^{b-1}n(e^{2\pi i f(n+1)}-e^{2\pi i f(n)})=be^{2\pi i f(b)}-ae^{2\pi i f(a)}-\sum_{n=a+1}^{b}e^{2\pi i f(n)}.$$

Subtracting, the desired result easily follows.

2.2. LEMMA 2.* *If $f(x)$ is real and has continuous first and second differential coefficients, and $f''(x)\geqslant r>0$ (or $\leqslant -r$) throughout the interval (a,b), then*

$$\int_a^b e^{2\pi i f(x)}\,dx=O\left(\frac{1}{\sqrt{r}}\right).$$

Since $f''(x)$ has the same sign throughout the range, $f'(x)$ is monotonic, and so vanishes at most once in the range—say at $x=c$. Then the integral is

$$\int_a^{c-\delta}\frac{d(e^{2\pi i f(x)})}{2\pi i f'(x)}+\int_{c-\delta}^{c+\delta}e^{2\pi i f(x)}\,dx+\int_{c+\delta}^{b}\frac{d(e^{2\pi i f(x)})}{2\pi i f'(x)}$$

$$=O\left(\frac{1}{|f'(c-\delta)|}\right)+O(\delta)+O\left(\frac{1}{|f'(c+\delta)|}\right),$$

by the second mean-value theorem (the first term being omitted if $c-\delta\leqslant a$, the last if $c+\delta\geqslant b$). Now

$$|f'(c+\delta)|=\left|\int_c^{c+\delta}f''(x)\,dx\right|\geqslant\delta r,$$

and similarly for $f'(c-\delta)$. Hence

$$\int_a^b e^{2\pi i f(x)}\,dx=O(\delta)+O\left(\frac{1}{\delta r}\right),$$

and the result follows on taking $\delta=1/\sqrt{r}$.

2.3. THEOREM 1.† *If $f(x)$ is real and has continuous first and second differential coefficients, and $f''(x)\geqslant r$ (or $\leqslant -r$) throughout the interval (a,b), and*

$$|f'(b)-f'(a)|=(b-a)R,$$

then

$$\sum_{a\leqslant n\leqslant b}e^{2\pi i f(n)}=O\left\{\frac{(b-a)R}{\sqrt{r}}\right\}+O\left(\frac{1}{\sqrt{r}}\right). \qquad (1)$$

We have $$R=\frac{1}{b-a}\left|\int_a^b f''(x)\,dx\right|\geqslant r,$$

* Van der Corput (1), 62. † Van der Corput (1), 62–3.

so that if $r \geqslant 1$ the result is trivial, the sum in question being obviously $O(b-a)$.

If $r<1$, divide the range of variation of $f'(x)$ into intervals $\nu-\frac{1}{2} \leqslant f'(x) < \nu+\frac{1}{2}$, where ν is an integer, or parts of such intervals. Let (a_ν, b_ν) be the corresponding x-intervals. Then

$$\sum_{a_\nu \leqslant n < b_\nu} e^{2\pi i f(n)} = \sum_{a_\nu \leqslant n < b_\nu} e^{2\pi i\{f(n)-\nu n\}}$$

$$= O\left\{ \int_{a_\nu}^{b_\nu} e^{2\pi i\{f(x)-\nu x\}}\, dx \right\} + O(1)$$

by Lemma 1, since $\left|\frac{d}{dx}\{f(x)-\nu x\}\right| \leqslant \frac{1}{2}$; and by Lemma 2 this is

$$O(1/\sqrt{r}) + O(1) = O(1/\sqrt{r}).$$

Since the whole variation of $f'(x)$ is $|f'(b)-f'(a)|$, the number of terms of the above form does not exceed $|f'(b)-f'(a)|+2$. This proves the theorem.

2.4. *Application to* $\zeta(\frac{1}{2}+it)$. It may be worth while noticing the result to which Theorem 1 leads in the problem of the order of $\zeta(\frac{1}{2}+it)$. We have*

$$\zeta(\tfrac{1}{2}+it) = \sum_{n<t} n^{-\frac{1}{2}-it} + O(1) \qquad (t>1). \tag{1}$$

Consider the sum

$$\sum_{n=N}^{N'} e^{-it\log n} \qquad (N < N' \leqslant 2N).$$

This is of the form considered, with $f(x) = -t\log x/2\pi$; hence

$$f''(x) = \frac{t}{2\pi x^2} \geqslant \frac{t}{8\pi N^2}$$

and $$\frac{f'(N')-f'(N)}{N'-N} = \frac{1}{N'-N}\left(\frac{t}{2\pi N} - \frac{t}{2\pi N'}\right) = \frac{t}{2\pi N N'} \leqslant \frac{t}{2\pi N^2}.$$

Hence Theorem 1 gives

$$\sum_{n=N}^{N'} e^{-it\log n} = O(\sqrt{t}) + O\left(\frac{N}{\sqrt{t}}\right) = O(\sqrt{t}),$$

if $N<t$. Hence by partial summation

$$\sum_{n=N}^{N'} n^{-\frac{1}{2}-it} = O\left(\frac{\sqrt{t}}{\sqrt{N}}\right) \qquad (N<t). \tag{2}$$

* See, for example, E. C. Titchmarsh, *The Zeta-function of Riemann* (Cambridge, 1930), Theorem 19.

Hence $$\sum_{\sqrt{t}\leqslant n<t} n^{-\frac{1}{2}-it} = \sum_{\sqrt{t}\leqslant n<2\sqrt{t}} + \sum_{2\sqrt{t}\leqslant n<4\sqrt{t}} + \dots$$

$$= O\left(\frac{\sqrt{t}}{\sqrt{(\sqrt{t})}}\right) + O\left(\frac{\sqrt{t}}{\sqrt{(2\sqrt{t})}}\right) + \dots = O(t^{\frac{1}{4}}).$$

Also $$\sum_{n<\sqrt{t}} n^{-\frac{1}{2}-it} = O\Big(\sum_{n<\sqrt{t}} n^{-\frac{1}{2}}\Big) = O(t^{\frac{1}{4}}).$$

Hence $$\zeta(\tfrac{1}{2}+it) = O(t^{\frac{1}{4}}).$$

Various methods which give substantially this result are known.*

3. LEMMA 3.† *If $f(x)$ is a real function, and ρ is a positive integer not exceeding $b-a$, then*

$$\left|\sum_{a\leqslant n\leqslant b} e^{2\pi i f(n)}\right| < A\frac{b-a}{\sqrt{\rho}} + A\left\{\frac{b-a}{\rho}\sum_{s=1}^{\rho-1}\left|\sum_{a\leqslant n\leqslant b-s} e^{2\pi i\{f(n+s)-f(n)\}}\right|\right\}^{\frac{1}{2}}.$$

Here $f(n)$, and so $e^{2\pi i f(n)}$, is originally defined for $a\leqslant n\leqslant b$. For convenience in the proof, let $e^{2\pi i f(n)}$ denote 0 if $n<a$ or $n>b$. Then

$$\sum_n e^{2\pi i f(n)} = \frac{1}{\rho}\sum_n\sum_{q=0}^{\rho-1} e^{2\pi i f(n+q)},$$

the inner sum vanishing if $n<a-\rho+1$ or $n>b$. Hence

$$\left|\sum_n e^{2\pi i f(n)}\right| \leqslant \frac{1}{\rho}\sum_n\left|\sum_{q=0}^{\rho-1} e^{2\pi i f(n+q)}\right| \leqslant \frac{1}{\rho}\left\{\sum_n 1\sum_n\left|\sum_{q=0}^{\rho-1} e^{2\pi i f(n+q)}\right|^2\right\}^{\frac{1}{2}};$$

and since there are at most $b-a+\rho\leqslant 2(b-a)$ values of n for which the inner sum does not vanish, this does not exceed

$$\frac{1}{\rho}\left\{2(b-a)\sum_n\left|\sum_{q=0}^{\rho-1} e^{2\pi i f(n+q)}\right|^2\right\}^{\frac{1}{2}}.$$

Now $$\left|\sum_{q=0}^{\rho-1} e^{2\pi i f(n+q)}\right|^2 = \sum_{q=0}^{\rho-1}\sum_{r=0}^{\rho-1} e^{2\pi i\{f(n+q)-f(n+r)\}}$$

$$= \rho + \sum_{r<q}\sum e^{2\pi i\{f(n+q)-f(n+r)\}} + \sum_{q<r}\sum e^{2\pi i\{f(n+q)-f(n+r)\}}.$$

Hence $$\sum_n\left|\sum_{q=0}^{\rho-1} e^{2\pi i f(n+q)}\right|^2 \leqslant 2(b-a)\rho + 2\left|\sum_n\sum_{r<q}\sum e^{2\pi i\{f(n+q)-f(n+r)\}}\right|.$$

In the last sum, $f(n+q)-f(n+r) = f(m+s)-f(m)$, for given values of m and s, $(0\leqslant s\leqslant \rho-1)$, just $\rho-s$ times, namely, $r=0$, $q=s$ up to $r=\rho-s-1$, $q=\rho-1$ (with a consequent value of n in each case).

* See E. C. Titchmarsh, loc. cit., Theorem 11 and the end of 2.21.

† Van der Corput (6), Satz I.

Hence the modulus of this sum is equal to

$$\Big|\sum_{s=1}^{\rho-1}(\rho-s)\sum_m e^{2\pi i\{f(m+s)-f(m)\}}\Big| \leqslant \rho\sum_{s=1}^{\rho-1}\Big|\sum_m e^{2\pi i\{f(m+s)-f(m)\}}\Big|.$$

Hence

$$\Big|\sum e^{2\pi i f(n)}\Big| \leqslant \frac{1}{\rho}\Big\{4(b-a)^2\rho+4(b-a)\rho\sum_{s=1}^{\rho-1}\Big|\sum_m e^{2\pi i\{f(m+s)-f(n)\}}\Big|\Big\}^{\frac{1}{2}},$$

and the result stated follows.

3.1. THEOREM 2. *If $f(x)$ is real and has continuous differential coefficients of the first three orders, and $f'''(x) \geqslant r_2$ (or $\leqslant -r_2$) throughout the interval (a,b), and* $|f''(b)-f''(a)| = (b-a)R_2$,

then

$$\sum_{a\leqslant n\leqslant b} e^{2\pi i f(n)} = O\Big\{\frac{(b-a)R_2^{\frac{1}{2}}}{r_2^{\frac{1}{2}}}\Big\} + O\Big\{\frac{(b-a)^{\frac{1}{2}}R_2^{\frac{1}{2}}}{r_2^{\frac{1}{2}}}\Big\}. \qquad (1)$$

We apply Theorem 1 to the sum

$$\sum_{a\leqslant n\leqslant b-s} e^{2\pi i g(n)},$$

where $g(x) = f(x+s)-f(x)$. We have

$$g''(x) = f''(x+s)-f''(x) = \int_x^{x+s} f'''(t)\,dt \geqslant sr_2 \quad (\text{or } \leqslant -sr_2),$$

so that we may take $r = sr_2$. Also

$$\begin{aligned} g'(b-s)-g'(a) &= \{f'(b)-f'(b-s)\}-\{f'(a+s)-f'(a)\} \\ &= sf''(\beta)-sf''(\alpha), \end{aligned}$$

where $b-s<\beta<b$, $a<\alpha<a+s$. Since $f''(x)$ is monotonic, it follows that

$$|g'(b-s)-g'(a)| \leqslant s|f''(b)-f''(a)|.$$

Hence

$$R = \Big|\frac{g'(b-s)-g'(a)}{b-s-a}\Big| \leqslant 2s\Big|\frac{f''(b)-f''(a)}{b-a}\Big| = 2sR_2,$$

provided that $s \leqslant \frac{1}{2}(b-a)$, which is true if $s<\rho \leqslant \frac{1}{2}(b-a)$.

With these conditions Theorem 1 gives

$$\sum_{a\leqslant n\leqslant b-s} e^{2\pi i g(n)} = O\Big\{\frac{(b-a)sR_2}{(sr_2)^{\frac{1}{2}}}\Big\} + O\Big\{\frac{1}{(sr_2)^{\frac{1}{2}}}\Big\}.$$

Hence, by Lemma 3,

$$\begin{aligned} \sum_{a\leqslant n\leqslant b} e^{2\pi i f(n)} &= O\Big(\frac{b-a}{\rho^{\frac{1}{2}}}\Big) + O\Big[\Big\{\frac{b-a}{\rho}\sum_{s=1}^{\rho-1}\Big(\frac{(b-a)s^{\frac{1}{2}}R_2}{r_2^{\frac{1}{2}}}+\frac{1}{s^{\frac{1}{2}}r_2^{\frac{1}{2}}}\Big)\Big\}^{\frac{1}{2}}\Big] \\ &= O\Big(\frac{b-a}{\rho^{\frac{1}{2}}}\Big) + O\Big[\Big\{\frac{(b-a)^2\rho^{\frac{1}{2}}R_2}{r_2^{\frac{1}{2}}}+\frac{b-a}{\rho^{\frac{1}{2}}r_2^{\frac{1}{2}}}\Big\}^{\frac{1}{2}}\Big] \\ &= O\Big(\frac{b-a}{\rho^{\frac{1}{2}}}\Big) + O\Big\{\frac{(b-a)\rho^{\frac{1}{4}}R_2^{\frac{1}{2}}}{r_2^{\frac{1}{2}}}\Big\} + O\Big\{\frac{(b-a)^{\frac{1}{2}}}{\rho^{\frac{1}{4}}r_2^{\frac{1}{4}}}\Big\}. \end{aligned}$$

The first two terms on the right are of the same form if $\rho = [r_2^{\frac{1}{2}} R_2^{-\frac{1}{2}}]$, and if this is an admissible value of ρ the result follows.

If $r_2^{\frac{1}{2}} R_2^{-\frac{1}{2}} < 1$, the first term on the right of (1) is of higher order than $b-a$, and the result is trivial. On the other hand, if $r_2^{\frac{1}{2}} R_2^{-\frac{1}{2}} > \frac{1}{2}(b-a)$, the result becomes trivial on account of the second term; for clearly $R_2 \geqslant r_2$, and hence

$$\frac{R_2^{\frac{1}{2}}}{r_2^{\frac{1}{2}}} \geqslant \frac{r_2^{\frac{1}{2}}}{R_2^{\frac{1}{2}}} > \{\tfrac{1}{2}(b-a)\}^{\frac{1}{2}}.$$

This proves the theorem.

3.2. *Application to* $\zeta(\frac{1}{2}+it)$. Defining $f(x)$ as in 2.4, we have $f'''(x) = -t/(\pi x^3)$. Hence for the sum

$$\sum_N^{N'} e^{2\pi i f(n)} \qquad (N < N' \leqslant 2N)$$

we have $r_2 = t/(8\pi N^3)$. Also

$$R_2 = \frac{1}{N'-N}\left|\frac{t}{2\pi N'^2} - \frac{t}{2\pi N^2}\right| = \frac{t(N+N')}{2\pi N^2 N'^2} \leqslant \frac{t}{\pi N^3}.$$

Hence, by Theorem 2,

$$\begin{aligned}\sum_N^{N'} e^{2\pi i f(n)} &= O\left\{N\left(\frac{t}{N^3}\right)^{\frac{1}{2}}\right\} + O\left\{N^{\frac{1}{2}}\left(\frac{t}{N^3}\right)^{-\frac{1}{2}}\right\} \\ &= O(N^{\frac{1}{2}}t^{\frac{1}{2}}) + O(Nt^{-\frac{1}{2}}).\end{aligned}$$

Hence, by partial summation,

$$\begin{aligned}\sum_N^{N'} n^{-\frac{1}{2}-it} &= O(t^{\frac{1}{2}}) + O(N^{\frac{1}{2}}t^{-\frac{1}{2}}) \\ &= O(t^{\frac{1}{2}}),\end{aligned}$$

if $N < t^{\frac{1}{3}}$. By 2.4 (2) this is true if $t^{\frac{1}{3}} \leqslant N < t$ also. Hence

$$\begin{aligned}\sum_{n<t} n^{-\frac{1}{2}-it} &= 1 + \sum_{n=2}^{3} + \ldots + \sum_{n=2^r}^{2^{r+1}-1} + \ldots \\ &= 1 + O(t^{\frac{1}{2}}) + O(t^{\frac{1}{2}}) + \ldots \\ &= O(t^{\frac{1}{2}} \log t),\end{aligned}$$

since there are $O(\log t)$ terms in the sum. Hence

$$\zeta(\tfrac{1}{2}+it) = O(t^{\frac{1}{2}} \log t).$$

4. We come now to the general theorem of which Theorems 1 and 2 are particular cases.

THEOREM 3. *Let k be a positive integer, and $f(x)$ a real function with continuous differential coefficients of the first $k+1$ orders; let $f^{(k+1)}(x) \geqslant r_k$ (or $\leqslant -r_k$) throughout the interval (a,b), and let*

$$|f^{(k)}(b) - f^{(k)}(a)| = (b-a)R_k.$$

Then

$$\sum_{a\leqslant n\leqslant b} e^{2\pi i f(n)} = O\left\{\frac{(b-a)R_k^{1/(2K-1)}}{r_k^{1/(4K-2)}}\right\} + O\left\{\frac{(b-a)^{1-1/K}R_k^{\alpha_k}}{r_k^{\alpha_k+1/(4K-2)}}\right\}, \qquad (1)$$

where $K=2^{k-1}$ *and* $0<\alpha_k<1$.

We do not give the actual value of α_k because it cannot be expressed in a very simple form, and because, in the applications, $R_k=O(r_k)$ and the terms involving α_k cancel.

If $k=1$ the theorem reduces to Theorem 1 ($\alpha_1=0$); if $k=2$ it reduces to Theorem 2 ($\alpha_2=\frac{1}{6}$). Supposing, then, that it holds for a particular value of k, we shall prove it for $k+1$.

Let $g(x)=f(x+s)-f(x)$. Then

$$g^{(k+1)}(x)=f^{(k+1)}(x+s)-f^{(k+1)}(x)=\int_x^{x+s} f^{(k+2)}(t)\,dt \geqslant sr_{k+1} \quad (\text{or} \leqslant -sr_{k+1}),$$

and for $g(x)$ we may take $r_k'=sr_{k+1}$. Also, as in the proof of Theorem 2, $R_k'\leqslant 2sR_{k+1}$. Hence the theorem applied to $g(x)$ gives

$$\sum_{a\leqslant n\leqslant b-s} e^{2\pi i g(n)} = O\left\{\frac{(b-a)(sR_{k+1})^{1/(2K-1)}}{(sr_{k+1})^{1/(4K-2)}}\right\} + O\left\{\frac{(b-a)^{1-1/K}(sR_{k+1})^{\alpha_k}}{(sr_{k+1})^{\alpha_k+1/(4K-2)}}\right\}$$

$$= O\left\{\frac{(b-a)s^{1/(4K-2)}R_{k+1}^{1/(2K-1)}}{r_{k+1}^{1/(4K-2)}}\right\} + O\left\{\frac{(b-a)^{1-1/K}R_{k+1}^{\alpha_k}}{s^{1/(4K-2)}r_{k+1}^{\alpha_k+1/(4K-2)}}\right\}.$$

Hence, by Lemma 3,

$$\sum_{a\leqslant n\leqslant b} e^{2\pi i f(n)} = O\left(\frac{b-a}{\rho^{\frac{1}{2}}}\right) +$$

$$+O\left[\left\{\frac{b-a}{\rho}\left(\frac{(b-a)\rho^{1+1/(4K-2)}R_{k+1}^{1/2(K-1)}}{r_{k+1}^{1/(4K-2)}}+\frac{(b-a)^{1-1/K}\rho^{1-1/(4K-2)}R_{k+1}^{\alpha_k}}{r_{k+1}^{\alpha_k+1/(4K-2)}}\right)\right\}^{\frac{1}{2}}\right]$$

$$= O\left(\frac{b-a}{\rho^{\frac{1}{2}}}\right) + O\left\{\frac{(b-a)\rho^{1/(8K-4)}R_{k+1}^{1/(4K-2)}}{r_{k+1}^{1/(8K-4)}}\right\} + O\left\{\frac{(b-a)^{1-1/(2K)}R_{k+1}^{\frac{1}{2}\alpha_k}}{r_{k+1}^{\frac{1}{2}\alpha_k+1/(8K-4)}\rho^{1/(8K-4)}}\right\}.$$

The first two terms on the right are of the same form if

$$\rho=[r_{k+1}^{1/(4K-1)}R_{k+1}^{-2/(4K-1)}],$$

and they are then of the form of the first term on the right of (1), with $k+1$ instead of k. The third term now becomes

$$(b-a)^{1-1/(2K)}R_{k+1}^{\alpha_{k+1}}/r_{k+1}^{\beta_{k+1}}$$

where

$$\alpha_{k+1}=\tfrac{1}{2}\alpha_k+\frac{1}{(4K-1)(4K-2)},\ \beta_{k+1}=\tfrac{1}{2}\alpha_k+\frac{1}{8K-4}+\frac{1}{(4K-1)(8K-4)};$$

and, since

$$\beta_{k+1}-\alpha_{k+1}=\frac{1}{8K-2},$$

this gives the second term on the right of (1), with $k+1$ instead of k.

This proves the theorem if the value chosen for ρ lies between 1 and $\frac{1}{2}(b-a)$. If $r_{k+1}^{1/(4K-1)}R_{k+1}^{-2/(4K-1)}<1$ the result is trivial, the first term on the right of (1), with $k+1$ instead of k, being of higher order than $b-a$. If

$$r_{k+1}^{1/(4K-1)}R_{k+1}^{-2/(4K-1)}>\tfrac{1}{2}(b-a),$$

then, since $R_{k+1}\geqslant r_{k+1}$,

$$\frac{R_{k+1}^{\alpha_{k+1}}}{r_{k+1}^{\alpha_{k+1}+1/(8K-2)}}\geqslant\frac{r_{k+1}^{1/(8K-2)}}{R_{k+1}^{1/(4K-1)}}>\{\tfrac{1}{2}(b-a)\}^{\frac{1}{2}},$$

and the second term in (1) is of higher order than $b-a$.

4.1. *Application to* $\zeta(s)$. Defining $f(x)$ as in 2.4 and 3.2, we have

$$f^{(k+1)}(x)=\frac{(-1)^{k+1}k!t}{2\pi x^{k+1}}.$$

Hence in the interval (N,N'), where $N'\leqslant 2N$, we have

$$r_k=\frac{k!t}{2^{k+2}\pi N^{k+1}},$$

and $$R_k=\left|\frac{f^{(k)}(N')-f^{(k)}(N)}{N'-N}\right|=\frac{1}{N'-N}\int_N^{N'}f^{(k+1)}(x)\,dx\leqslant\frac{k!t}{2\pi N^{k+1}}.$$

Hence Theorem 3 gives

$$\sum_{n=N}^{N'}e^{2\pi i f(n)}=O\left\{N\left(\frac{t}{N^{k+1}}\right)^{1/(4K-2)}\right\}+O\left\{N^{1-1/K}\left(\frac{t}{N^{k+1}}\right)^{-1/(4K-2)}\right\}.$$

The second term may be omitted if

$$N\leqslant t^{K/(kK-K+1)},$$

and with this condition we have, by partial summation,

$$\sum_{n=N}^{N'}n^{-\sigma-it}=O(N^{1-(k+1)/(4K-2)-\sigma}t^{1/(4K-2)}). \tag{1}$$

Let q be an integer not less than 2, $Q=2^{q-1}$, and let

$$\sigma=1-\frac{q+1}{4Q-2}.$$

We have to select the values of k best suited to this value of σ and the value of N. First take $k=q$. Then (1) gives

$$\sum_{n=N}^{N'}n^{-\sigma-it}=O(t^{1/(4Q-2)})\qquad(N\leqslant t^{Q/(qQ-Q+1)}), \tag{2}$$

and adding $O(\log t)$ terms of this type, as in 3.2, we obtain

$$\sum_{n\leqslant t^{Q/(qQ-Q+1)}} n^{-\sigma-it} = O(t^{1/(4Q-2)}\log t). \tag{3}$$

Next, taking any value of k less than q, we obtain

$$\sum_{n=N}^{N'} n^{-\sigma-it} = O(N^{(q+1)/(4Q-2)-(k+1)/(4K-2)}t^{1/(4K-2)}).$$

If
$$t^{\frac{2K}{(k+1)2K-2K+1}} \leqslant N \leqslant t^{\frac{K}{kK-K+1}}, \tag{4}$$

it follows that

$$\sum_{n=N}^{N'} n^{-\sigma-it} = O\left(t^{\frac{2K}{(k+1)2K-2K+1}\left(\frac{q+1}{4Q-2}-\frac{k+1}{4K-2}\right)+\frac{1}{4K-2}}\right). \tag{5}$$

The index of t in (5) does not exceed that in (3), if

$$\frac{2K}{2kK+1}\left(\frac{q+1}{4Q-2}-\frac{k+1}{4K-2}\right)+\frac{1}{4K-2} \leqslant \frac{1}{4Q-2},$$

and this reduces to $\quad Q-K \geqslant (q-k)K.$

Writing $k=q-r$, so that $r>0$, this is

$$(2r-1)K \geqslant rK,$$

which is true. We therefore obtain the same result as in (3), but with n ranging over the interval given by (4). Adding the results for $k=q,\ q-1,\ldots,\ 1$, we obtain

$$\sum_{n<t} n^{-\sigma-it} = O(t^{1/(4Q-2)}\log t),$$

and (4) of § 1 follows.

5. In conclusion we shall consider briefly the circle problem. Let $R(x)$ be the number of solutions in integers of the inequality $u^2+v^2 \leqslant x$, and let

$$R_1(x) = \int_0^x R(y)\,dy.$$

Then it is known* that

$$R_1(x) = \tfrac{1}{2}\pi x^2 + \frac{x}{\pi}\sum_{n=1}^{\infty}\frac{r(n)}{n}J_2\{2\pi\sqrt{(nx)}\}$$

where $r(n)$ is the number of solutions in integers of $u^2+v^2=n$. Hence, if $0<\alpha<1$,

$$\int_x^{x+x^\alpha} \{R(y)-\pi y\}\,dy$$
$$= \frac{x+x^\alpha}{\pi}\sum_{n=1}^{\infty}\frac{r(n)}{n}J_2\{2\pi\sqrt{(nx+nx^\alpha)}\} - \frac{x}{\pi}\sum_{n=1}^{\infty}\frac{r(n)}{n}J_2\{2\pi\sqrt{(nx)}\}$$

* See e.g. Landau's *Vorlesungen über Zahlentheorie*, ii. 204–6.

$$= \sum_{n=1}^{N} \frac{r(n)}{\sqrt{n}} \int_{x}^{x+x^{\alpha}} \sqrt{y} J_1\{2\pi\sqrt{(ny)}\}\, dy + \frac{x+x^{\alpha}}{\pi} \sum_{n=N+1}^{\infty} \frac{r(n)}{n} J_2\{2\pi\sqrt{(nx+nx^{\alpha})}\} -$$

$$- \frac{x}{\pi} \sum_{n=N+1}^{\infty} \frac{r(n)}{n} J_2\{2\pi\sqrt{(nx)}\}$$

$$= \Sigma_1 + \Sigma_2 + \Sigma_3,$$

say. Now

$$J_1\{2\pi\sqrt{(ny)}\} = \frac{\sin\{2\pi\sqrt{(ny)} - \frac{1}{4}\pi\}}{\pi(ny)^{\frac{1}{4}}} + O\left\{\frac{1}{(ny)^{\frac{3}{4}}}\right\}.$$

Hence

$$\Sigma_1 = O\left\{\int_{x}^{x+x^{\alpha}} |\phi(y)| y^{\frac{1}{4}}\, dy\right\} + O(x^{\alpha-\frac{1}{4}})$$

where

$$\phi(y) = \sum_{n=1}^{N} \frac{r(n)}{n^{\frac{3}{4}}} e^{2\pi i\sqrt{(ny)}}.$$

Similarly,

$$\Sigma_3 = O\{x^{\frac{3}{4}} |\psi(x)|\} + O(x^{\frac{1}{4}})$$

where

$$\psi(x) = \sum_{n=N+1}^{\infty} \frac{r(n)}{n^{\frac{3}{4}}} e^{2\pi i\sqrt{(nx)}},$$

and a similar result holds for Σ_2.

5.1. Consider $\phi(y)$. We have

$$\phi(y) = 2 \sum_{\substack{\mu^2+\nu^2 \leqslant N \\ \mu<\nu}} \sum \frac{e^{2\pi i\sqrt{\{(\mu^2+\nu^2)y\}}}}{(\mu^2+\nu^2)^{\frac{3}{4}}} + \sum_{2\nu^2 \leqslant N} \frac{e^{2\pi i\nu\sqrt{n}}}{2^{\frac{3}{4}}\nu^{\frac{3}{2}}}$$

$$= 2 \sum_{\mu \leqslant \sqrt{(\frac{1}{2}N)}} \sum_{\mu<\nu\leqslant\sqrt{(N-\mu^2)}} \frac{e^{2\pi i\sqrt{\{(\mu^2+\nu^2)y\}}}}{(\mu^2+\nu^2)^{\frac{3}{4}}} + O(1),$$

and we have to apply Theorem 3 to the sum

$$\sum_{\nu=a}^{b} e^{2\pi i\sqrt{\{(\mu^2+\nu^2)y\}}} \qquad (a<b\leqslant 2a).$$

Here

$$f^{(\nu)} = (\mu^2+\nu^2)^{\frac{1}{4}} y^{\frac{1}{2}},$$

$$f''(\nu) = \frac{\mu^2\sqrt{y}}{(\mu^2+\nu^2)^{\frac{3}{4}}}, \ldots \qquad f^{(4)}(\nu) = \frac{15\mu^2\nu(3\mu^2-4\nu^2)\sqrt{y}}{(\mu^2+\nu^2)^{9/2}}, \ldots$$

and it is not difficult to see that we have generally

$$|f^{(k+1)}(\nu)| > C\mu^2 y^{\frac{1}{2}} \nu^{-k-2} \qquad (\mu<\nu)$$

where C depends on k only, though actually we shall require this for $k=4$ only. Hence for $a \leqslant \nu \leqslant b \leqslant 2a$ we take $r_k = C\mu^2 y^{\frac{1}{2}} a^{-k-2}$; and similarly $R_k = C\mu^2 y^{\frac{1}{2}} a^{-k-2}$. Hence

$$\sum_{\nu=a}^{b} e^{2\pi i\sqrt{\{(\mu^2+\nu^2)y\}}} = O\{a(\mu^2 y^{\frac{1}{2}} a^{-k-2})^{1/(4K-2)}\} + O\{a^{1-1/K}(\mu^2 y^{\frac{1}{2}} a^{-k-2})^{-1/(4K-2)}\}.$$

Hence, by partial summation,

$$\sum_{\nu=a}^{b} \frac{e^{2\pi i\sqrt{\{(\mu^2+\nu^2)y\}}}}{(\mu^2+\nu^2)^{\frac{3}{4}}} = O(a^{-\frac{1}{2}-(k+2)/(4K-2)}\mu^{1/(2K-1)}y^{1/(8K-4)}) + \\ + O(a^{-\frac{1}{2}-1/K+(k+2)/(4K-2)}\mu^{-1/(2K-1)}y^{-1/(8K-4)})$$

for $\mu<\nu$, $b\leqslant 2a$. Hence

$$\sum_{\mu<\nu\leqslant\sqrt{(N-\mu^2)}} \frac{e^{2\pi i\sqrt{\{(\mu^2+\nu^2)y\}}}}{(\mu^2+\nu^2)^{\frac{3}{4}}} = \sum_{\nu=\mu+1}^{2\mu} + \sum_{2\mu+1}^{4\mu} + \dots \\ = O(\mu^{-\frac{1}{2}-k/(4K-2)}y^{1/(8K-4)}) + O(\mu^{-\frac{1}{2}-1/K+k/(4K-2)}y^{-1/(8K-4)}).$$

Hence, if $k>1$,

$$\phi(y) = O(N^{\frac{1}{4}-k/(8K-4)}y^{1/(8K-4)}) + O(N^{\frac{1}{4}-1/(2K)+k/(8K-4)}y^{-1/(8K-4)}).$$

Hence

$$\sum\nolimits_1 = O(N^{\frac{1}{4}-k/(8K-4)}x^{\alpha+\frac{1}{4}+1/(8K-4)}) + O(N^{\frac{1}{4}-1/(2K)+k/(8K-4)}x^{\alpha+\frac{1}{4}-1/(8K-4)}), \quad (1)$$

the term $O(x^{\alpha-\frac{1}{4}})$ being absorbed in the last term written.

5.2. A similar argument applies to $\psi(x)$. We have

$$\psi(x) = 2\sum_{\substack{\mu^2+\nu^2>N\\ \mu<\nu}}\sum \frac{e^{2\pi i\sqrt{\{(\mu^2+\nu^2)x\}}}}{(\mu^2+\nu^2)^{\frac{3}{4}}} + \sum_{2\nu^2>N} \frac{e^{2\pi i\nu\sqrt{x}}}{2^{\frac{3}{2}}\nu^{\frac{3}{2}}} \\ = 2\sum_{\mu\leqslant\sqrt{(\frac{1}{2}N)}}\sum_{\nu>\sqrt{(N-\mu^2)}} \frac{e^{2\pi i\sqrt{\{(\mu^2+\nu^2)x\}}}}{(\mu^2+\nu^2)^{\frac{3}{4}}} + 2\sum_{\mu>\sqrt{(\frac{1}{2}N)}}\sum_{\nu>\mu} \dots + O(N^{-\frac{1}{2}}).$$

Proceeding as before, we obtain

$$\sum_{\nu>\sqrt{(N-\mu^2)}} \frac{e^{2\pi i\sqrt{\{(\mu^2+\nu^2)x\}}}}{(\mu^2+\nu^2)^{\frac{3}{4}}} = O(N^{-\frac{1}{2}-(k+2)/(8K-4)}\mu^{1/(2K-1)}x^{1/(8K-4)}) + \\ + O(N^{-\frac{1}{2}-1/(2K)+(k+2)/(8K-4)}\mu^{-1/(2K-1)}x^{-1/(8K-4)}),$$

and hence

$$\sum_{\mu\leqslant\sqrt{(\frac{1}{2}N)}}\sum_{\nu>\sqrt{(N-\mu^2)}} \frac{e^{2\pi i\sqrt{\{(\mu^2+\nu^2)x\}}}}{(\mu^2+\nu^2)^{\frac{3}{4}}} = O(N^{-\frac{1}{2}-k/(8K-4)}x^{1/(8K-4)}) + \\ + O(N^{-\frac{1}{2}-1/(2K)+k/(8K-4)}x^{-1/(8K-4)}).$$

It is easily seen that the same result is obtained for the other part of $\psi(x)$. Hence

$$\sum\nolimits_3 = O(N^{-\frac{1}{2}-k/(8K-4)}x^{\frac{3}{4}+1/(8K-4)}) + O(N^{-\frac{1}{2}-1/(2K)+k/(8K-4)}x^{\frac{3}{4}-1/(8K-4)}), \quad (1)$$

and a similar result holds for $\sum_2$.

5.3. The first term on the right of 5.1(1) is of the same form as the first term on the right of 5.2(1) if

$$N = [x^{1-2\alpha}].$$

With this value of N we obtain

$$\int_x^{x+x^\alpha} \{R(y)-\pi y\}\,dy = O(x^p)+O(x^q),$$

where

$$p = \tfrac{1}{2}+\tfrac{1}{2}\alpha-\frac{k-2k\alpha-1}{8K-4}, \qquad q = \tfrac{1}{2}+\tfrac{1}{2}\alpha-\frac{1-2\alpha}{K}+\frac{k-2k\alpha-1}{8K-4}.$$

Since $R(y)$ is non-decreasing it follows that

$$x^\alpha\{R(x)-\pi(x+x^\alpha)\} < O(x^p)+O(x^q),$$
$$R(x)-\pi x < \pi x^\alpha+O(x^{p-\alpha})+O(x^{q-\alpha}).$$

The first two terms on the right are of the same form if $p = 2\alpha$, i.e.

$$\alpha = \frac{4K-k-1}{12K-2k-6}.$$

This is least when $k = 4$, $K = 8$, and is then equal to $\frac{27}{82}$. But then $q-\alpha = 24\frac{1}{2}/82$, so that the last term in the inequality can be omitted, and

$$R(x)-\pi x < Ax^{\frac{27}{82}}.$$

By considering integrals over $(x-x^\alpha, x)$ a similar inequality is obtained on the other side. This proves (2) of § 1.

REFERENCES

J. G. van der Corput:

1. 'Zahlentheoretische Abschätzungen': *Math. Annalen*, 84 (1921), 53–79.
2. 'Verschärfung der Abschätzung beim Teilerproblem': *Math. Annalen*, 87 (1922), 39–65.
3. 'Neue zahlentheoretische Abschätzungen, erste Mitteilung': *Math. Annalen*, 89 (1923), 215–54.
4. 'Zum Teilerproblem': *Math. Annalen*, 98 (1928), 697–716.
5. 'Zahlentheoretische Abschätzungen, mit Anwendung auf Gitterpunktprobleme': *Math. Zeitschrift*, 28 (1928), 301–10.
6. 'Neue zahlentheoretische Abschätzungen, zweite Mitteilung': *Math. Zeitschrift*, 29 (1929), 397–426.

J. G. van der Corput and J. F. Koksma, (1) 'Sur l'ordre de grandeur de la fonction $\zeta(s)$ de Riemann dans la bande critique': *Annales de Toulouse* (3) 22 (1930), 1–39.

L. W. Nieland, (1) 'Zum Kreisproblem': *Math. Annalen*, 98 (1928), 717–36.

A. Walfisz, (1) 'Zur Abschätzung von $\zeta(\frac{1}{2}+it)$': *Göttinger Nachrichten* (1924), 155–8.

ELECTRICAL NOTES

IV. ALTERNATING CURRENTS IN NETWORKS

By F. B. PIDDUCK

[Received 20 April 1931]

THIS note originated in an inquiry by my pupil Mr. E. S. Jackson as to how far one is justified in assigning inductances to the arms of a network and potentials to the corners. The difficulties are mentioned somewhat timidly by Gray, *Absolute Measurements in Electricity and Magnetism*, 2nd ed., pp. 241–2. The potentials are unnecessary and are eliminated by grouping the arms into closed circuits; so we have first to consider the self and mutual inductances of the *circuits* and define them if possible for all frequencies of alternation. Though practically important networks need no great refinement, there is a certain interest in finding correct equations, for example, for a Wheatstone's bridge made up of six plain wires soldered together. We shall work from the field equations

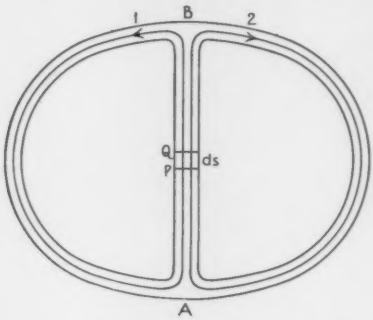
$$4\pi\mathbf{u} = \operatorname{curl}\mathbf{H}, \quad -\partial\mathbf{H}/\partial t = c\operatorname{curl}\mathbf{E}, \quad \operatorname{div}\mathbf{E} = \operatorname{div}\mathbf{H} = 0, \quad \mathbf{u} = \sigma c\mathbf{E}$$

or the corresponding circuital relations, as in the theory of the distribution of current in cylindrical conductors.

Consider a single loop of wire in which current is induced by a magnetic field varying sinoidally. The lines of flow or current-filaments are fixed curves, and $\int u\,ds/\sigma = -dN/dt$ for each, where u is the current-density, σ the conductivity and N the flux of magnetic induction through the filament. Let u and N be complex, containing a time factor $e^{\lambda t}$ with purely imaginary index. The time mean of the product of the real parts of u and N is $\frac{1}{4}(u^*N+uN^*)$, and similarly for other complexes. If dS is the area of cross-section of a filament, the total current in the wire is $i = \int u\,dS$ and the circuital equation gives $\int uu^*ds\,dS/\sigma = -\lambda\int u^*N\,dS$. Define the resistance of the loop so that the time mean $\frac{1}{2}\int uu^*ds\,dS/\sigma$ of the rate of development of heat is $\frac{1}{2}Rii^*$ in accordance with the usual definition of effective value, and the mean flux $\bar{N}$ through the different filaments to be $\int u^*N\,dS/\int u^*\,dS$. Then $Rii^* = -\lambda i^*\bar{N}$, or $d\bar{N}/dt + Ri = 0$, the law of electromagnetic induction for the whole circuit.

Let AB be an arm of a network in which two circuits coalesce, and let u_1, u_2 be the separate current-densities. Since electrons entering the arm from either side become mixed in a small fraction of a milli-

metre, u_1 is proportional to u_2 at all points and the constituent current filaments interpenetrate in strict proportion. If i is the total current, we have for the first circuit $(\sum Ri)i_1^* = -\lambda \int u_1^* N_1 \, dS_1$, where summation is over all the arms in the circuit. We can approximate for thin wires by extending Maxwell's theory of geometrical mean distance. Let P, Q be two sections of AB distant ds apart, and to fix ideas let dS_1, dS_2 be the sectional areas at P of two typical filaments of circuits 1, 2 respectively. The flux through the whole filament 1 due to the Ampèrean elementary force of the part of filament 2 between P and Q is approximately $ds(-2\log r + C)u_2 dS_2$



when the distance r between dS_1 and dS_2 is small, and C depends on the position of P. The contribution to $\int u_1^* N_1 \, dS_1$ of that part of circuit 2 between P and Q is therefore

$$ds \int (-2\log r + C)u_1^* u_2 \, dS_1 dS_2 = ds(-2\log \bar{r} + C)i_2^* i_2,$$

where
$$\log \bar{r} = \int \log r \, u_1^* u_2 \, dS_1 dS_2 \Big/ \int u_1^* u_2 \, dS_1 dS_2.$$

Since u_1, u_2 in this last equation can be taken from the total distribution of flow in the arm, $\bar{r}$ is real. It follows that

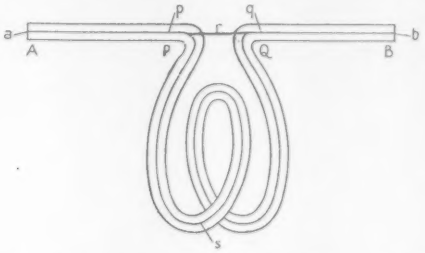
$$d(\text{external } \bar{N}_1)/dt + L_1 di_1/dt + M di_2/dt + \sum Ri = 0,$$

and the circuit equations of any network are correct, if every self and mutual inductance is calculated by replacing the two circuits concerned by the locus of the centroids of their cross-sections where they are distinct, and by two filaments at the distance $\bar{r}$ defined above for each cross-section at which they coalesce. At a circular section of diameter $2a$,

$$\log \bar{r} = \log a - (\text{ber}\,\xi \,\text{ber}'\xi + \text{bei}\,\xi \,\text{bei}'\xi)/\xi(\text{ber}'^2\xi + \text{bei}'^2\xi),$$

where $\xi^2 = 8\pi n/10^7\rho$, n being the frequency and ρ the static resistance of the wire in ohms per metre. The limitations of the theory are that the total length of wire occupied by junctions and points of sharp curvature and rapid change of section must be small.

The electromagnetic scalar and vector potentials V, $\mathbf{A}$ both vary as $e^{\lambda t}$, and $\partial\mathbf{A}/\partial t + c\mathbf{E}$ is the negative gradient of cV. Let an arm AB



contain a nearly re-entrant coil PQ of many turns, and let $apsqb$ be one of its filaments. Then if prq is a short line joining p to q,

$$cV_a - cV_b = \frac{d}{dt}\int_{aprqb} \mathbf{A}\,d\mathbf{s} + \frac{d}{dt}\int_{psqrp} \mathbf{A}\,d\mathbf{s} + \int_{apsqb} c\mathbf{E}\,d\mathbf{s}.$$

The first term on the right is by hypothesis much smaller than the second. Multiply by u^*dS and integrate over the cross-section. Then if $V_A = \int u^* V_a\,dS / \int u^*\,dS$, etc., $cV_A - cV_B = d\bar{N}/dt + Ri$, where $\bar{N}$ is the mean flux through PQ when closed by a short length of wire. The eliminated potentials are the values of the scalar electromagnetic potential at the corners. There are no potentials when current is induced from outside in a network of resistances and inductances, any more than in a conducting sheet. A condenser whose plates are far apart requires an ether, but one with infinitesimal separation of the plates falls within the present theory and appears as a doublet singularity of V, which satisfies Laplace's equation elsewhere.

ASYMPTOTIC PARTITION FORMULAE

I. PLANE PARTITIONS

By E. MAITLAND WRIGHT

[Received 16 May 1931]

1. THE idea of the ordinary partition of an integer has been considered by many writers, with or without restrictions on the size of the parts or on the number of parts. Let us confine our attention for the moment to unrestricted partitions, and regard a partition as involving a definite arrangement of the parts in descending* order of magnitude along a line; thus, a partition of 8 is

$$431. \qquad (1.01)$$

Then we may regard this as a one-dimensional or line partition.

This point of view leads us to an obvious generalization to two-dimensional or plane partitions, which may be defined as follows.

Take any line partition of the integer n and arrange the parts in rows and columns, so that a descending* order of magnitude is in evidence in each row from left to right, and in each column from top to bottom. Each row starts from the first column and each column from the first row. Such an arrangement is termed a plane partition of n.

Thus, the particular plane partitions of 8 which are derived from the line partition (1.01) are

$$\left.\begin{array}{llll} 431 & 43 & 41 & 4 \\ & 1 & 3 & 3 \\ & & & 1 \end{array}\right\} \qquad (1.02)$$

The idea of a plane partition was introduced by MacMahon. He deals with the general problem of such partitions, with or without restrictions on one or more of the following:

(i) the size of each part,

(ii) the number of rows,

(iii) the number of columns.

He obtains the generating function† for each of the various cases by means of intricate and beautiful analysis based on his theory of Lattice Functions.

* 'Descending order of magnitude' is used in the wide sense.

† *Combinatory Analysis* (Camb., 1916), vol. ii, pp. 173–243.

The case 'each part not greater than unity' is equivalent to that of unrestricted line partitions. Here the generating function is

$$\prod_{l=1}^{\infty}(1-x^l)^{-1} = 1+\sum_{n=1}^{\infty} p(n)x^n. \tag{1.03}$$

Hardy and Ramanujan* obtained an asymptotic formula for $p(n)$ with an error only $O(n^{-\frac{1}{4}})$.

Of the other cases the unrestricted case is the most difficult and interesting from the asymptotic point of view. For this case the generating function is

$$f(x) = \prod_{l=1}^{\infty}(1-x^l)^{-l} = 1+\sum_{n=1}^{\infty} q(n)x^n, \tag{1.04}$$

where $q(n)$ is the number of plane partitions of n defined as above. In this paper I shall obtain an asymptotic formula for $q(n)$ for large n.

At first sight, the analogy between (1.03) and (1.04) suggests the possibility of some simple proof of (1.04) on the lines of Euler's 'intuitive' proof of (1.03). I have been unable to find any such proof; and as from one of MacMahon's memoirs† it is clear that he devoted much attention to the problem of proving (1.04), it seems unlikely that a simple proof exists.

2.1. I use the following scheme of notation:

$\zeta(s)$ is the zeta-function of Riemann, and B_s is the sth Bernoullian number; that is,

$$\zeta(s) = \sum_{m=1}^{\infty}\frac{1}{m^s},$$

and

$$B_s = 4s\int_0^{\infty}\frac{y^{2s-1}\,dy}{e^{2\pi y}-1} = \frac{4s\Gamma(2s)\zeta(2s)}{(2\pi)^{2s}}, \tag{2.11}$$

$$A = \zeta(3). \qquad c = 2\int_0^{\infty}\frac{y\log y\,dy}{e^{2\pi y}-1}, \tag{2.12}$$

$$\alpha_s = \frac{B_sB_{s+1}}{4s(s+1)\Gamma(2s+1)} = \frac{2\Gamma(2s+2)\zeta(2s)\zeta(2s+2)}{s(2\pi)^{4s+2}}; \tag{2.13}$$

β_s $(s \leqslant r+1)$ is the coefficient of y^s in the expansion of

$$\exp(-\sum_{l=1}^{r+1}\alpha_l y^l)$$

* *Proc. London Math. Soc.* (2) 17 (1918), 75–115. The small order of the error in this case is due to special properties of the generating function. The results of the present paper indicate the improbability of anything similar being true in the case of $q(n)$.

† *Phil. Trans. Royal Society*, 211A, 77.

in ascending powers of y. It is obvious that, so long as $s \leqslant r+1$, β_s is independent of r. Also, if $|y|$ be small,

$$\exp(-\sum_{l=1}^{r+1} \alpha_l y^l) = \sum_{s=0}^{r+1} \beta_s y^s + O(|y|^{r+2}); \tag{2.14}$$

$b_{s,m}$ is the coefficient of y^{2m} in the expansion of

$$(1+y)^{2s+2m+13/12}(3+2y)^{-m-\frac{1}{2}} \tag{2.15}$$

in ascending powers of y.

H and K are positive numbers whose values vary from one occurrence to another. H is an absolute constant (such as 3), while K depends on r only. By $|u(n)| = \big(Ov(n)\big)$, I mean that, for any fixed r,

$$|u(n)| < Kv(n)$$

for all n; and by $|u(n)| = o\big(v(n)\big)$, I mean that, for any $\epsilon > 0$ and any fixed r,

$$\frac{|u(n)|}{v(n)} < \epsilon$$

for all n greater than some fixed $n_0 = n_0(\epsilon, r)$.

N is a large positive number, to be assigned later, and C_N is the circle

$$|x| = e^{-1/N}$$

in the complex x-plane.

θ is the principal value of $\arg x$,

$$z = \log\frac{1}{x} = \log\left|\frac{1}{x}\right| - i\theta = \rho e^{i\phi},$$

$$w = \Re\left(\frac{\pi}{2z}\right) = \frac{\pi\cos\phi}{2\rho},$$

$$C_N = C_N' + C_N'',$$

where C_N' is that part of C_N on which $|\theta| < 1/N$.

2.2. The object of this paper is to prove the following

THEOREM. *The asymptotic expansion* of $q(n)$ is*

$$\frac{(2^{25}A^7)^{1/36}e^c}{2\pi^{\frac{1}{2}}n^{25/36}}\exp(3\,.\,2^{-\frac{2}{3}}A^{\frac{1}{3}}n^{\frac{2}{3}})\left\{1+\sum_{h=1}^{r}\frac{\gamma_h}{n^{\frac{1}{2}h}}+O(n^{-\frac{1}{2}(r+1)})\right\}, \tag{2.21}$$

where

$$\gamma_h = \frac{1}{\sqrt{\pi}}\left(\frac{4}{A}\right)^{\frac{1}{2}h}\sum_{s=0}^{h}\{(-)^{h-s}\beta_s b_{s,\,h-s}\Gamma(h-s+\tfrac{1}{2})A^s\}. \tag{2.22}$$

From the product form of $f(x)$ in (1.04), we see that $f(x)$ is an analytic function regular within the unit circle, every point of the

* The numerical values of the constants involved in the first term of the expansion are given at the end of the paper.

circumference being an essential singularity. Then, by Cauchy's theorem, $q(n)$ may be expressed in the form

$$\frac{1}{2\pi i}\int_{C_N}\frac{f(x)\,dx}{x^{n+1}}; \tag{2.23}$$

since C_N encloses the origin and lies entirely within the unit circle.

It will appear that, to put it roughly, $x=1$ is much the 'heaviest' singularity of $f(x)$; and I shall show that, when N is large, the dominant terms in $q(n)$ are supplied by the portion of the integral (2.23) on C'_N. For this purpose, we need two lemmas with regard to the value of $f(x)$ on C'_N, C''_N respectively.

3. Lemma I. *When x is on C'_N, we have*

$$\left|f(x)-e^{c}z^{1/12}\exp\left(\frac{A}{z^2}\right)\left\{\sum_{s=0}^{r+1}\beta_s z^{2s}\right\}\right|<KN^{-2r-4-1/12}e^{AN^2}. \tag{3.01}$$

This is proved by using Cauchy's theorem in a manner which may be adapted to obtain the asymptotic or exact transformation of many similar functions.

We note that for x on C'_N, we have

$$\rho\leqslant\sqrt{2}/N \text{ and } |\phi|<\tfrac{1}{4}\pi.$$

We understand by E a number whose absolute value is less than

$$Ke^{-H/\rho}.$$

It is obvious that, since $\cos\phi>1/\sqrt{2}$,

$$e^{-(H/\rho)\cos\phi}<e^{-H/\rho}=E,$$

for x on C'_N. Also

$$\rho^{\alpha}(\log\rho)^{\beta}e^{-H/\rho}=E,$$

if α and β are numbers (positive or negative) depending only on r.

3.1. Suppose that p is a positive integer and δ a small positive number. Let Γ_p be a contour in the t-plane running from $t=\delta$ to $t=p+\frac{1}{2}$, coincident with the real axis except near $t=1, 2,\ldots, p$, where we pass round small semicircles above these points. Let Δ be the quarter-circle with centre at the origin running from $i\delta$ to δ. Finally, let Γ'_p, Δ' be the reflections of Γ_p, Δ in the real axis.

We take that value of $\log(1-e^{-tz})$ which tends to zero as $t\to+\infty$ along the real axis, and note that the only singularities of

$$\psi(t)=\frac{t\log(1-e^{-tz})}{1-e^{-2\pi it}}, \tag{3.11}$$

apart from the origin, are simple poles at $t = \pm l$, and logarithmic singularities at $t = \pm 2l\pi i/z$, where l runs through all positive integral values.

We draw two straight lines, one through $t = iw$, parallel to the real axis, and the other through $t = p+\frac{1}{2}$, parallel to the imaginary axis. These two straight lines form an indented rectangle with Δ, Γ_p, and the imaginary axis from $i\delta$ to iw. By Cauchy's theorem, since the integrand has no singularities within the rectangle,

$$\int_{\Gamma_p} + \int_{iw}^{i\delta} \psi(t)\,dt = -\int_{\Delta} + \int_{iw}^{p+\frac{1}{2}+iw} + \int_{p+\frac{1}{2}+iw}^{p+\frac{1}{2}} \psi(t)\,dt = I_1+I_2+I_3. \quad (3.12)$$

The reader will readily find that, for $|\phi| < \frac{1}{4}\pi$,

$$|I_1| < H(\delta \log\delta + \delta|\log z|),$$

and

$$|I_3| < H\left(p+\frac{\pi}{2\rho}\right)e^{-p\rho/\sqrt{2}}.$$

Now let δ tend to zero and p to infinity. The limits of integration in I_2 are now iw and $iw+\infty$; and Γ_p becomes $\Gamma_\infty = \Gamma$ (say). The equation (3.12) becomes

$$\int_{\Gamma} \psi(t)\,dt + \int_{iw}^{0} \psi(t)\,dt = I_2. \quad (3.13)$$

Let us now consider I_2. Making the transformation $v = tz$, we find that

$$|I_2| < He^{-(\pi^2/\rho)\cos\phi} \int_{iw}^{\infty+iw} |t\log(1-e^{-tz})|\,|dt| = EI(\phi), \quad (3.14)$$

where

$$I(\phi) = \int_{L} |v|\,|\log(1-e^{-v})|\,|dv|,$$

and L is the half straight line in the v-plane from

$$v = \tfrac{1}{2}\pi i e^{i\phi}\cos\phi$$

inclined at an angle ϕ to the positive direction of the real axis. Divide L into L_1 and L_2 so that, on L, $\Re(v) \leqslant \frac{1}{2}\pi$. Then L_1 is of bounded length, and the integrand on L_1 is uniformly bounded for $|\phi| \leqslant \frac{1}{4}\pi$. Also, on L_2, $\Re(v) > H|v|$, and so we have

$$\int_{L_2} |v|\,|\log(1-e^{-v})|\,|dv| < H\int_{L_2} |ve^{-v}|\,|dv| < H\int_{L_2} |v|\,e^{-H|v|}\,|dv| < H.$$

Hence $I(\phi) < H$, and we have $I_2 = E$.

(3.13) now becomes

$$\int_{\Gamma} \frac{t\log(1-e^{-tz})}{1-e^{-2\pi it}}\,dt = \int_{0}^{iw} \frac{t\log(1-e^{-tz})}{1-e^{-2\pi it}}\,dt + E. \tag{3.15}$$

By similar reasoning, we have also

$$\int_{\Gamma'} \frac{t\log(1-e^{-tz})}{e^{2\pi it}-1}\,dt = \int_{0}^{-iw} \frac{t\log(1-e^{-tz})}{e^{2\pi it}-1}\,dt + E. \tag{3.16}$$

By Cauchy's theorem we know that

$$\begin{aligned}
&\int_{\Gamma'} \frac{t\log(1-e^{-tz})}{e^{2\pi it}-1}\,dt - \int_{\Gamma} \frac{t\log(1-e^{-tz})}{1-e^{-2\pi it}}\,dt \\
&\quad = \sum_{l=1}^{\infty} l\log(1-e^{-lz}) + \int_{\Gamma} \left(\frac{t\log(1-e^{-tz})}{e^{2\pi it}-1} - \frac{t\log(1-e^{-tz})}{1-e^{-2\pi it}}\right) dt \\
&\quad = -\log f(x) - \int_{0}^{\infty} t\log(1-e^{-tz})\,dt \\
&\quad = -\log f(x) + \frac{A}{z^2}.
\end{aligned} \tag{3.17}$$

Finally, from (3.15), (3.16), and (3.17), we have

$$\log f(x) = \frac{A}{z^2} + \int_{0}^{iw} \frac{t\log(1-e^{-tz})}{1-e^{-2\pi it}}\,dt - \int_{0}^{-iw} \frac{t\log(1-e^{-tz})}{e^{2\pi it}-1}\,dt + E,$$

which reduces to

$$\log f(x) = \frac{A}{z^2} + 2\int_{0}^{w} \frac{y\log(2\sin\frac{1}{2}yz)}{e^{2\pi y}-1}\,dy + E. \tag{3.18}$$

3.2. We now proceed to calculate the value of

$$2\int_{0}^{w} \frac{y\log(2\sin\frac{1}{2}yz)}{e^{2\pi y}-1}\,dy. \tag{3.21}$$

From the well-known formula

$$\sin\tau = \tau\prod_{m=1}^{\infty}\left(1-\frac{\tau^2}{m^2\pi^2}\right),$$

we have at once, for $|\tau| < \pi$,

$$\log\sin\tau = \log\tau - \sum_{s=1}^{\infty}\sum_{m=1}^{\infty} \frac{\tau^{2s}}{sm^{2s}\pi^{2s}} = \log\tau - \sum_{s=1}^{\infty} \frac{\zeta(2s)\tau^{2s}}{s\pi^{2s}}. \tag{3.22}$$

If we now write $\tau = \frac{1}{2}yz$, we have, on the range of integration in (3.21),

$$|\tau| = |\tfrac{1}{2}yz| \leqslant \tfrac{1}{2}w|z| \leqslant \tfrac{1}{4}\pi.$$

The series (3.22) is uniformly convergent for such values of τ, and so we may write

$$2\int_0^w \frac{y\log(2\sin\frac{1}{2}yz)}{e^{2\pi y}-1}\,dy$$

$$=2\int_0^w \frac{y\log(yz)\,dy}{e^{2\pi y}-1}-2\Big(\sum_{s=1}^{r+1}+\sum_{s=r+2}^{\infty}\Big)\frac{\zeta(2s)z^{2s}}{s(2\pi)^{2s}}\int_0^w \frac{y^{2s+1}\,dy}{e^{2\pi y}-1}$$

$$=I_4+S_1+S_2. \tag{3.23}$$

Now we have

$$I_4=2\int_0^\infty \frac{y\log(yz)}{e^{2\pi y}-1}\,dy+O(\rho^{-1}e^{-(\pi^2/\rho)\cos\phi}\log\rho)$$

$$=2\int_0^\infty \frac{y\log y}{e^{2\pi y}-1}\,dy+2\log z\int_0^\infty \frac{y\,dy}{e^{2\pi y}-1}+E$$

$$=c+\tfrac{1}{12}\log z+E, \tag{3.24}$$

since

$$\int_0^\infty \frac{y\,dy}{e^{2\pi y}-1}=\tfrac{1}{4}B_1=\tfrac{1}{24}.$$

Also, by (2.11) and (2.13),

$$-S_1=2\sum_{s=1}^{r+1}\frac{\zeta(2s)z^{2s}}{s(2\pi)^{2s}}\int_0^\infty \frac{y^{2s+1}\,dy}{e^{2\pi y}-1}+E$$

$$=\sum_{s=1}^{r+1}\alpha_s z^{2s}+E. \tag{3.25}$$

Finally,

$$|S_2|<K\rho^{2r+4}\int_0^w \frac{y^{2r+5}}{e^{2\pi y}-1}\Big\{\sum_{s=0}^{\infty}\Big(\frac{y\rho}{2\pi}\Big)^{2s}\Big\}\,dy$$

$$=K\rho^{2r+4}\int_0^w \frac{y^{2r+5}\,dy}{\Big\{1-\Big(\frac{y\rho}{2\pi}\Big)^2\Big\}(e^{2\pi y}-1)}$$

$$<K\rho^{2r+4}\int_0^\infty \frac{y^{2r+5}\,dy}{e^{2\pi y}-1}<K\rho^{2r+4}. \tag{3.26}$$

Combining (3.18), (3.24), (3.25), and (3.26), we see that the error terms may all be included in one term, since

$$E<Ke^{-H/\rho}<K\rho^{2r+4}.$$

Hence,

$$\log f(x) = \frac{A}{z^2} + \frac{1}{12}\log z + c - \sum_{s=0}^{r+1} \alpha_s z^{2s} + O(\rho^{2r+4});$$

and from this we have

$$f(x) = e^c z^{1/12} e^{A/z^2} \left\{ \sum_{s=0}^{r+1} \beta_s z^{2s} + O(\rho^{2r+4}) \right\}.$$

Now, on C_N'', $\rho \leqslant \sqrt{2}/N$, and

$$\left| \exp\left(\frac{A}{z^2}\right) \right| \leqslant \exp\left|\frac{A}{z^2}\right| \leqslant e^{AN^2};$$

so we have

$$\left| f(x) - e^c z^{1/12} e^{A/z^2} \left\{ \sum_{s=0}^{r+1} \beta_s z^{2s} \right\} \right| < KN^{-2r-4-1/12} e^{AN^2},$$

the result of Lemma I.

4. Lemma II. *Given any $\epsilon > 0$, we can find an $N_0 = N_0(\epsilon)$, such that, for all $N > N_0$ and all x on C_N'', we have*

$$|f(x)| < \exp\{(A - \tfrac{1}{2} + \epsilon)N^2\}.$$

The proof of this is very simple. We have

$$\log f(x) = -\sum_{l=1}^{\infty} l \log(1-x^l) = \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \frac{l x^{ml}}{m} = \sum_{m=1}^{\infty} \frac{x^m}{m(1-x^m)^2};$$

and so

$$|\log f(x)| \leqslant \frac{|x|}{|1-x|^2} + \sum_{m=2}^{\infty} \frac{|x|^m}{m(1-|x^m|)^2}$$

$$= \sum_{m=1}^{\infty} \frac{|x|^m}{m(1-|x^m|)^2} - \left\{ \frac{|x|}{(1-|x|)^2} - \frac{|x|}{|1-x|^2} \right\}.$$

But

$$\sum_{m=1}^{\infty} \frac{|x|^m}{m(1-|x^m|)^2} = \log f(|x|) = AN^2 + o(N^2),$$

by Lemma I. Also

$$\frac{|x|}{(1-|x|)^2} - \frac{|x|}{|1-x|^2} = \frac{|x|}{(1-|x|)^2} \left\{ 1 - \left(\frac{1-|x|}{|1-x|} \right)^2 \right\} \geqslant \tfrac{1}{2} N^2 + o(N^2),$$

since

$$\frac{1-|x|}{|1-x|} \leqslant \frac{1}{\sqrt{2}} + O\left(\frac{1}{N}\right) \quad \text{for } |\theta| \geqslant \frac{1}{N}.$$

So we have finally

$$|\log f(x)| \leqslant (A - \tfrac{1}{2})N^2 + o(N^2) < (A - \tfrac{1}{2} + \epsilon)N^2,$$

if $N > N_0(\epsilon)$.

5.1. I now put $N=(n/2A)^{\frac{1}{2}}$, so that $n=2AN^3$. It is convenient to work mainly in terms of N, rather than n. We have

$$q(n)=\frac{1}{2\pi i}\int_{C_N}\frac{f(x)\,dx}{x^{n+1}}$$

$$=\frac{1}{2\pi i}\Big(\int_{C_N'}+\int_{C_N''}\Big)=J_1+J_2.$$

By Lemma I,

$$J_1=\frac{1}{2\pi i}\int_{(1-i)/N}^{(1+i)/N}f(e^{-z})e^{2AN^3z}\,dz$$

$$=\frac{e^c}{2\pi i}\int_{(1-i)/N}^{(1+i)/N}\Big(\sum_{s=0}^{r+1}\beta_s z^{2s+1/12}\Big)\exp\Big(\frac{A}{z^2}+2AN^3z\Big)\,dz+$$

$$+O(N^{-2r-5-1/12}e^{3AN^2})$$

$$=e^c\sum_{s=0}^{r+1}\frac{\beta_s P_s}{N^{2s+1+1/12}}+O(N^{-2r-5-1/12}e^{3AN^2}),$$

where

$$P_s=\frac{N^{2s+1+1/12}}{2\pi i}\int_{(1-i)/N}^{(1+i)/N}z^{2s+1/12}\exp\Big(\frac{A}{z^2}+2AN^3z\Big)\,dz$$

$$=\frac{1}{2\pi i}\int_{1-i}^{1+i}v^{2s+1/12}\exp\Big\{AN^2\Big(2v+\frac{1}{v^2}\Big)\Big\}\,dv,$$

if we put $v=Nz$.

5.2. Our next step is to obtain an asymptotic expansion of P_s in descending powers of N^2 by the method of 'steepest descents'.*

The curve of steepest descent through $v=1$ is found by putting

$$\Im(2v+v^{-2})=0.$$

If we write $v=X+iY$, and reject the factor Y, we find the equation of the curve $\mathfrak{C}$, namely,

$$(X^2+Y^2)^2=X. \tag{5.21}$$

This is the equation of a closed curve touching $X=1$ at $(1,0)$, and $X=0$ at $(0,0)$, and cutting $X=-Y$, $X=Y$ at the points $D\,(2^{-\frac{1}{2}},-2^{-\frac{1}{2}})$, $E\,(2^{-\frac{1}{2}},2^{-\frac{1}{2}})$ respectively. $\mathfrak{C}$ lies entirely in the strip $0\leqslant X\leqslant 1$. The figure shows the general shape of the curve.

* Watson, *Proc. London Math. Soc.* (2) 17 (1918), 117, gives references to memoirs discussing this method.

We consider the v-plane as cut from 0 to $-\infty$ along the real axis, and we take that value of $v^{1/12}$ which is real and positive at $v=1$. We regard the contour $\mathfrak{C}$ as starting from 0 and being described in the counter-clockwise sense.

Let us write

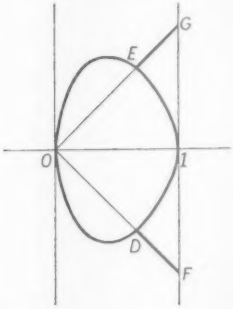
$$\xi_s(v)=\frac{v^{2s+1/12}}{2\pi i}\exp\left\{AN^2\left(2v+\frac{1}{v^2}\right)\right\},$$

and

$$\overline{P}_s=\int_{\mathfrak{C}}\xi_s(v)\,dv.$$

If F, G are the points $1-i$, $1+i$, we have

$$\Re(v^{-2})=\frac{X^2-Y^2}{(X^2+Y^2)^2}\leqslant 0$$



on the straight lines DF, EG and the parts OD, EO of $\mathfrak{C}$. Hence, on these contours, since $X<1$, $|v|$ is bounded, and $s\leqslant r+1$, we have

$$|\xi_s(v)|=\frac{|v|^{2s+1/12}}{2\pi}e^{2AN^2X}\leqslant Ke^{2AN^2}.$$

It follows that

$$|P_s-\overline{P}_s|\leqslant\left|\int_D^F\xi_s(v)\,dv\right|+\left|\int_G^E\right|+\left|\int_E^O\right|+\left|\int_O^F\right|<Ke^{2AN^2}. \tag{5.22}$$

In $\overline{P}_s$, we now put

$$t=-i\left(\frac{v-1}{v}\right)(2v+1)^{\frac{1}{2}}, \tag{5.23}$$

that value of the square root being taken which is positive at $v=1$. Then

$$t^2=3-2v-v^{-2},$$

and $\mathfrak{C}$ transforms into the real axis in the t-plane. So we have

$$\overline{P}_s=e^{3AN^2}\int_{-\infty}^{+\infty}\chi_s(t)e^{-AN^2t^2}\,dt,$$

where
$$\chi_s(t) = \frac{v^{2s+1/12}}{2\pi i}\frac{dv}{dt}.$$

5.3.* Near $v = 1$, we can expand (5.23) in a power series

$$t = \sum_{m=1}^{\infty} d_m (v-1)^m,$$

where $d_1 = -i\sqrt{3}$. Since $d_1 \neq 0$, by reversion of the series we have

$$v = 1 + \sum_{1}^{\infty} g_m t^m,$$

and
$$\frac{dv}{dt} = \sum_{1}^{\infty} m g_m t^m,$$

each with a positive radius of convergence. Hence we have

$$\chi_s(t) = \sum_{m=0}^{\infty} a_m t^m, \tag{5.31}$$

the series having a radius of convergence greater than some K.

Since
$$t^2 = 3 - 2v - v^{-2},$$

we see that
$$t\frac{dt}{dv} = \frac{1-v^3}{v^3};$$

and so
$$\frac{dv}{dt} = \frac{v^3 t}{1-v^3} = \frac{iv^2(2v+1)^{\frac{1}{2}}}{1+v+v^2}.$$

Then, on $\mathfrak{C}$,
$$\left|\frac{v^{2s+1/12}}{2\pi i}\frac{dv}{dt}\right| < K;$$

and $|\chi_s(t)|$ is therefore bounded on the whole of the real axis in the t-plane.

Since (5.31) holds on an interval of the real axis near $t = 0$, we have, on the whole of the real axis,

$$\left|\chi_s(t) - \sum_{m=0}^{2r+3} a_m t^m\right| < K|t|^{2r+4}.$$

Hence
$$\overline{P}_s = \int_{-\infty}^{+\infty} \chi_s(t) e^{-AN^2t^2}\, dt$$

$$= \sum_{m=0}^{2r+3} \int_{-\infty}^{+\infty} a_m t^m e^{-AN^2t^2}\, dt + M_r,$$

where
$$|M_r| < K\int_{-\infty}^{+\infty} t^{2r+4} e^{-AN^2t^2}\, dt < \frac{K}{N^{2r+5}}.$$

* I am indebted to Watson's paper (loc. cit., pp. 133, 137) for the method I use in this paragraph.

Now $$\int_{-\infty}^{\infty} t^m e^{-AN^2t^2}\,dt = 0,$$

if m is odd; while if m is even,

$$\int_{-\infty}^{\infty} t^m e^{-AN^2t^2}\,dt = \frac{\Gamma(\frac{1}{2}m+\frac{1}{2})}{(AN^2)^{\frac{1}{2}(m+1)}}.$$

So finally we have the asymptotic expansion

$$\begin{aligned} P_s &= \overline{P}_s + O(e^{2AN^2}) \\ &= e^{3AN^2}\sum_{m=0}^{r+1}\frac{a_{2m}\Gamma(m+\frac{1}{2})}{(AN^2)^{m+\frac{1}{2}}} + O\left(\frac{e^{3AN^2}}{N^{2r+5}}\right). \end{aligned} \tag{5.32}$$

5.4. To calculate a_{2m} we proceed as follows. Near $t=0$,

$$\chi_s(t) = \sum_{m=0}^{\infty} a_m t^m,$$

and so $$a_{2m} = \frac{1}{2\pi i}\int^{(0+)} \chi_s(t)\frac{dt}{t^{2m+1}},$$

the integral being taken along a small loop around $t=0$.

Then $$\begin{aligned} a_{2m} &= -\frac{1}{4\pi^2}\int^{(1+)} \frac{v^{2s+1/12}}{t^{2m+1}}\,dv \\ &= \frac{(-1)^m}{4\pi^2 i}\int^{(1+)} \frac{v^{2s+1/12+2m+1}}{(v-1)^{2m+1}(2v+1)^{m+\frac{1}{2}}}\,dv. \end{aligned}$$

Putting $v = 1+y$, we have

$$\begin{aligned} a_{2m} &= \frac{(-1)^m}{4\pi^2 i}\int^{(0+)} \frac{(1+y)^{2s+2m+1+1/12}\,dy}{y^{2m+1}(3+2y)^{m+\frac{1}{2}}} \\ &= \frac{(-1)^m b_{s,m}}{2\pi}, \qquad \text{by (2.15).} \end{aligned}$$

5.5. By Lemma II,

$$\begin{aligned} |J_2| &< \exp\{(A-\tfrac{1}{2}+\epsilon)N^2+nN^{-1}\} \\ &= \exp\{(3A-\tfrac{1}{2}+\epsilon)N^2\} < K(N^{-2r-5-1/12}e^{3AN^2}), \end{aligned}$$

if we take $\epsilon < \frac{1}{2}$. Combining this with (5.32), we have

$$\begin{aligned} q(n) = J_1+J_2 &= e^c\sum_{s=0}^{r+1}\frac{\beta_s P_s}{N^{2s+13/12}} + O\left(\frac{e^{3AN^2}}{N^{2r+5+1/12}}\right) \\ &= \frac{e^c}{2\pi}e^{3AN^2}\sum_{s=0}^{r+1}\sum_{m=0}^{r+1}\frac{(-1)^m\beta_s b_{s,m}\Gamma(m+\frac{1}{2})}{A^{m+\frac{1}{2}}N^{2(m+s+1)+1/12}} + \\ &\qquad + O(N^{-2r-5-1/12}e^{3AN^2}). \end{aligned}$$

Let us now put $h = m+s$, and sum with respect to h and s. Then s runs from 0 to h; and we may collect all terms for which $h > r$ in one error term. We have

$$q(n) = \frac{e^c e^{3AN^2}}{2\pi A^{\frac{1}{2}} N^{25/12}} \left\{ \sum_{h=0}^{r} \left(\frac{A}{4}\right)^{\frac{1}{2}h} \frac{\pi^{\frac{1}{2}} \gamma_h}{A^h N^{2h}} \right\} + O\left(\frac{e^{3AN^2}}{N^{2r+49/12}}\right).$$

If we now write $N = (n/2A)^{\frac{1}{2}}$, we have the result of the theorem.

6. It is clear that the method I have used could be extended to determine the asymptotic expansion of the nth coefficient of other functions of a similar type. The essential step in each case is the proof of two lemmas corresponding to those proved here.

The next paper of this series will be concerned with the different idea of 'weighted' partitions, and, in particular, with the coefficients of the function

$$\prod_{l=1}^{\infty} (1-ax^l)^{-1}.$$

The case $a > 1$ seems of the greater interest, but the appropriate method is entirely distinct from the one used here. The case $a < 1$, however, which is of less interest in itself, provides a simple example of an application of the method of this paper.

Note. The numerical evaluation of the number c involves that of the series

$$\sum_{m=2}^{\infty} \frac{\log m}{m^2} = 0{\cdot}9375482\ldots.$$

We know* that

$$A = \zeta(3) = 1{\cdot}20205690\ldots,$$

and so we have

$$\frac{(2^{25}A^7)^{1/36} e^c}{2\pi^{\frac{1}{2}}} = 0{\cdot}4009989\ldots.$$

We have also, for the coefficient of $n^{\frac{1}{2}}$ in the index of the exponent, the value

$$3{\cdot}2^{-\frac{1}{2}} A^{\frac{1}{2}} = 2{\cdot}0094\ldots.$$

The values of $q(n)$ for $n = 1, 2, \ldots, 16$ have been calculated by Macmahon, and appear in the right-hand column of the table on p. 332, vol. ii, of his *Combinatory Analysis*. In this table, $q(n)$ appears as the number of 'solid graphs' of n nodes, but the two definitions are equivalent.

* Knopp, *Infinite Series*, p. 561.

AN ABSOLUTE PARTIAL DIFFERENTIAL CALCULUS

By H. S. RUSE

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1. Introduction

In the Absolute Differential Calculus* of Ricci and Levi-Civita, a tensor is a set of functions, obeying certain laws of transformation, of a *single* set of variables. The present paper outlines a theory of tensor-functions of *two* independent sets of variables. In a certain sense the ordinary tensor theory is analogous to the elementary differential calculus of functions of a single variable, while the following work is comparable with the theory of the partial differentiation of functions of two independent variables. It is true that the formation of a covariant derivative in the Absolute Differential Calculus involves partial differentiation with respect to each one of the given set of variables, but it is the peculiar nature of the subject that the process is thought of as a single operation and is usually symbolized by the addition of one subscript to the operand, just as the derivative of a function $f(x)$ in the elementary calculus is often represented by the same symbol with an added accent: that is, by $f'(x)$.

Now although on the one hand the suffix notation for covariant derivatives is adhered to in this paper, yet, on the other, there are two kinds of subscripts (bracketed and unbracketed), corresponding to covariant differentiation with respect to one or other of the two given sets of variables. This necessity of introducing two notations is, from the point of view here adopted, analogous to the similar need in the elementary differential calculus, in which the partial derivatives of a function $f(x, y)$ of two independent variables are often represented by $f_x(x, y)$ and $f_y(x, y)$.

It was these considerations which led to the choice of the title of this paper.

There is no theoretical reason why the ideas presented below should not be extended to tensor-functions of any number of sets of independent variables, instead of two only. There is, however, a practical reason, for such an extension would involve algebra of an unsatis-

* This is referred to below as 'the ordinary tensor theory'.

factorily elaborate character. The notation and terminology of this paper, based as they are on those of the ordinary tensor theory, are already complicated enough without further generalization.

The subject is treated from an analytical standpoint, although § 5 contains references to geometry. The earlier paragraphs are devoted mainly to definitions, which in themselves contain the essence of the paper. In the concluding paragraph appear a few remarks concerning the general aspect of the theory.

2. Preliminary definitions

Let $(x^\mu) \equiv (x^1, x^2, \ldots, x^n)$ and $(\bar{x}^i) \equiv (\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^m)$ be two sets of independent variables. It may be stated at once that Greek suffixes will always refer to the x's (and will therefore take the values $1, 2, \ldots, n$) and Latin to the $\bar{x}$'s, taking the values $1, 2, \ldots, m$.

Suppose that we have n independent functions $f_\mu(x^\nu)$ of the (x^μ), but *not* of the $(\bar{x}^i)$, and m independent functions $\phi_i(\bar{x}^j)$ of the $(\bar{x}^i)$, but *not* of the (x^μ). We can then define two new sets of variables $(x'^\mu) \equiv (x'^1, x'^2, \ldots, x'^n)$ and $(\bar{x}'^i) \equiv (\bar{x}'^1, \bar{x}'^2, \ldots, \bar{x}'^m)$ by the equations

$$x'^\mu = f_\mu(x^\nu), \qquad (\mu = 1, 2, \ldots, n), \tag{1}$$

and

$$\bar{x}'^i = \phi_i(\bar{x}^j), \qquad (i = 1, 2, \ldots, m). \tag{2}$$

Now let $T^{\mu(p)}$ denote the array

$$\begin{pmatrix} T^{1(1)} & T^{1(2)} & \ldots & T^{1(m)} \\ T^{2(1)} & T^{2(2)} & \ldots & T^{2(m)} \\ \ldots & \ldots & \ldots & \ldots \\ T^{n(1)} & T^{n(2)} & \ldots & T^{n(m)} \end{pmatrix}$$

where $T^{1(1)}$, $T^{1(2)}, \ldots$ are each functions, mn in number, of both the (x^μ) and the $(\bar{x}^i)$.

Further, let

(i) $T'^{\mu(p)}$ be a set of mn functions of the (x'^μ) and $(\bar{x}^i)$;

(ii) ${}'T^{\mu(p)}$ be a set of mn functions of the (x^μ) and $(\bar{x}'^i)$;

(iii) ${}'T'^{\mu(p)}$ be a set of mn functions of the (x'^μ) and $(\bar{x}'^i)$.

Then, if these sets of functions are related by the equations

$$T'^{\mu(p)} = \frac{\partial x'^\mu}{\partial x^\alpha} T^{\alpha(p)} \qquad (p = 1, 2, \ldots, m), \tag{3}$$

$${}'T^{\mu(p)} = \frac{\partial \bar{x}'^p}{\partial \bar{x}^a} T^{\mu(a)} \qquad (\mu = 1, 2, \ldots, n), \tag{4}$$

$${}'T'^{\mu(p)} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial \bar{x}'^p}{\partial \bar{x}^a} T^{\alpha(a)} \qquad \begin{pmatrix} p = 1, 2, \ldots, m \\ \mu = 1, 2, \ldots, n \end{pmatrix}, \tag{5}$$

the repetitions of the suffixes α and a implying summations from 1 to n and from 1 to m respectively, we shall say that $T^{\mu(p)}$ are *the components of a contravariant double-vector* in the systems (x^μ) and $(\bar{x}^i)$, and that it has components $T'^{\mu(p)}$ in the systems (x'^μ) and $(\bar{x}^i)$, components ${}'T^{\mu(p)}$ in the systems (x^μ) and $(\bar{x}'^i)$, and components ${}'T'^{\mu(p)}$ in the systems (x'^μ) and $(\bar{x}'^i)$; or alternatively, $T^{\mu(p)}$ will be called *a contravariant x-vector and a contravariant $\bar{x}$-vector.**

From this definition it is evident that, if the $(\bar{x}^i)$ are kept constant, then $T^{\mu(1)}$, $T^{\mu(2)}$,..., $T^{\mu(m)}$ are m contravariant vectors in the sense of the ordinary tensor theory. A similar remark applies if the (x^μ) are kept constant and the $(\bar{x}^i)$ allowed to vary. The indices μ and p are therefore ordinary tensor-suffixes, μ belonging to the x's and p to the $\bar{x}$'s. For greater clearness the tensor-suffixes belonging to the $\bar{x}$'s will be placed in brackets in addition to being written in Latin characters.

More generally, a set of $m^{r+s}n^{\rho+\sigma}$ functions $T^{\lambda_1\lambda_2\ldots\lambda_\rho(p_1p_2\ldots p_r)}_{\mu_1\mu_2\ldots\mu_\sigma(q_1q_2\ldots q_s)}$ of the x's and $\bar{x}$'s will be said to constitute a *double-tensor* if they transform according to the laws:

I. $$T'^{\lambda_1\lambda_2\ldots\lambda_\rho(p_1p_2\ldots p_r)}_{\mu_1\mu_2\ldots\mu_\sigma(q_1q_2\ldots q_s)} = \frac{\partial x'^{\lambda_1}}{\partial x^{\alpha_1}}\frac{\partial x'^{\lambda_2}}{\partial x^{\alpha_2}}\cdots\frac{\partial x'^{\lambda_\rho}}{\partial x^{\alpha_\rho}}\frac{\partial x^{\beta_1}}{\partial x'^{\mu_1}}\frac{\partial x^{\beta_2}}{\partial x'^{\mu_2}}\cdots\frac{\partial x^{\beta_\sigma}}{\partial x'^{\mu_\sigma}}T^{\alpha_1\alpha_2\ldots\alpha_\rho(p_1\ldots p_r)}_{\beta_1\beta_2\ldots\beta_\sigma(q_1\ldots q_s)}$$

for a transformation of the (x^μ) to the (x'^μ) with the $(\bar{x}^i)$ unaltered;

II. $${}'T^{\lambda_1\lambda_2\ldots\lambda_\rho(p_1p_2\ldots p_r)}_{\mu_1\mu_2\ldots\mu_\sigma(q_1q_2\ldots q_s)} = \frac{\partial\bar{x}'^{p_1}}{\partial\bar{x}^{a_1}}\frac{\partial\bar{x}'^{p_2}}{\partial\bar{x}^{a_2}}\cdots\frac{\partial\bar{x}'^{p_r}}{\partial\bar{x}^{a_r}}\frac{\partial\bar{x}^{b_1}}{\partial\bar{x}'^{q_1}}\cdots\frac{\partial\bar{x}^{b_s}}{\partial\bar{x}'^{q_s}}T^{\lambda_1\ldots\lambda_\rho(a_1a_2\ldots a_r)}_{\mu_1\ldots\mu_\sigma(b_1b_2\ldots b_s)}$$

for a transformation of the $(\bar{x}^i)$ to the $(\bar{x}'^i)$ with the (x^μ) unaltered;

III. $${}'T'^{\lambda_1\ldots\lambda_\rho(p_1\ldots p_r)}_{\mu_1\ldots\mu_\sigma(q_1\ldots q_s)} = \frac{\partial x'^{\lambda_1}}{\partial x^{\alpha_1}}\cdots\frac{\partial x'^{\lambda_\rho}}{\partial x^{\alpha_\rho}}\frac{\partial x^{\beta_1}}{\partial x'^{\mu_1}}\cdots\frac{\partial x^{\beta_\sigma}}{\partial x'^{\mu_\sigma}}\frac{\partial\bar{x}'^{p_1}}{\partial\bar{x}^{a_1}}\cdots\frac{\partial\bar{x}'^{p_r}}{\partial\bar{x}^{a_r}}\frac{\partial\bar{x}^{b_1}}{\partial\bar{x}'^{q_1}}\cdots\frac{\partial\bar{x}^{b_s}}{\partial\bar{x}'^{q_s}}T^{\alpha_1\ldots\alpha_\rho(a_1\ldots a_r)}_{\beta_1\ldots\beta_\sigma(b_1\ldots b_s)}$$

for a simultaneous transformation of the (x^μ) and of the $(\bar{x}^i)$. When greater explicitness is required, $T^{\lambda_1\ldots\lambda_\rho(p_1\ldots p_r)}_{\mu_1\ldots\mu_\sigma(q_1\ldots q_s)}$ will be described as *a mixed x-tensor of rank $\rho+\sigma$ and a mixed $\bar{x}$-tensor of rank $r+s$.*

In particular, if a set of functions of the (x^μ) and of the $(\bar{x}^\mu)$ behave like scalars for transformations of one of the sets of variables, and like the components of a vector or tensor for transformations of the other set, they will be described as forming a *scalar-vector* or *scalar-*

* More briefly, a double-vector is a set of functions of the x's and $\bar{x}$'s which transforms according to (3) for transformations of the x's with the $\bar{x}$'s unaltered, according to (4) when the $\bar{x}$'s are transformed but not the x's, and according to (5) for a simultaneous transformation of both sets of variables.

tensor as the case may be. Thus, for example, a set of n functions T_μ transforming according to the laws

$$T'_\mu = \frac{\partial x^\alpha}{\partial x'^\mu} T_\alpha \quad [(\bar{x}^i) \text{ unaltered}],$$

$$'T_\mu = T_\mu \quad [(x^\mu) \text{ unaltered}],$$

$$'T'_\mu = \frac{\partial x^\alpha}{\partial x'^\mu} T_\alpha \quad [\text{simultaneous transformation}],$$

are the components of a scalar-vector, or, more explicitly, of *a covariant x-vector and an $\bar{x}$-scalar.**

Similarly, a contravariant x-vector and a covariant $\bar{x}$-tensor such as $T^\mu_{(pq)}$ will sometimes be called a *vector-tensor*; and so on.

An accent placed after a functional symbol will always, as above, denote the system obtained by transforming from the (x^μ) to the (x'^μ) with the $(\bar{x}^i)$ unaltered; one placed *before* the symbol will indicate a transformation from the $(\bar{x}^i)$ to the $(\bar{x}'^i)$ with the (x^μ) unaltered; and one before and one after will represent the system obtained by a simultaneous transformation of both sets of variables.

It should not be necessary to show that the definitions given above are self-consistent, or that the transformation-laws possess the reciprocal and transitive properties, for the proofs would be very similar to the corresponding theory in the ordinary tensor calculus. Moreover, it is evident that sums of like double-tensors are double-tensors of the same kind, that products of double-tensors are double-tensors, and so on. Obvious properties of this kind will be assumed without comment in the subsequent paragraphs.

3. Scalar-connexions

A *scalar-connexion* is defined to be a set of functions of the x's and $\bar{x}$'s which behave like scalars for transformations of one set of variables and like the components of an affine connexion for transformations of the other set.

For example, let $\Gamma^\lambda_{\mu\nu}$ be a set of n^3 functions of both sets of variables which transform according to the laws:

$$\left.\begin{aligned} \Gamma'^\lambda_{\mu\nu} &= \frac{\partial x'^\lambda}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x'^\mu}\frac{\partial x^\gamma}{\partial x'^\nu}\Gamma^\alpha_{\beta\gamma} + \frac{\partial x'^\lambda}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\nu} \\ {}'\Gamma^\lambda_{\mu\nu} &= \Gamma^\lambda_{\mu\nu} \\ {}'\Gamma'^\lambda_{\mu\nu} &= \Gamma'^\lambda_{\mu\nu} \end{aligned}\right\} \qquad (6)$$

* From the point of view of the ordinary tensor theory there are of course n of these scalars, namely, $T_1, T_2, \ldots, T_n$.

for transformations of the variables of the types indicated by the accents. Then the $\Gamma^{\lambda}_{\mu\nu}$ are the components of a scalar-connexion, or, as we shall sometimes say, of an *x-connexion and $\bar{x}$-scalar.*

Similarly, a set of m^3 functions L^{p}_{qr} such that

$$\left.\begin{aligned} L'^{p}_{qr} &= L^{p}_{qr} \\ {}'L^{p}_{qr} &= \frac{\partial \bar{x}'^{p}}{\partial \bar{x}^{a}} \frac{\partial \bar{x}^{b}}{\partial \bar{x}'^{q}} \frac{\partial \bar{x}^{c}}{\partial \bar{x}'^{r}} L^{a}_{bc} + \frac{\partial \bar{x}'^{p}}{\partial \bar{x}^{a}} \frac{\partial^2 \bar{x}^{a}}{\partial \bar{x}'^{q} \partial \bar{x}'^{r}} \\ {}'L'^{p}_{qr} &= {}'L^{p}_{qr} \end{aligned}\right\} \tag{7}$$

constitute, by definition, an *x-scalar and an $\bar{x}$-connexion.*

4. Covariant derivatives

Let $T^{\mu(p)}_{\nu(q)}$ be a double-tensor of the type indicated by the suffixes. For the sake of simplicity a double-tensor possessing only one contravariant and one covariant suffix of each kind is considered, the formulae being easily generalized to double-tensors of any rank.

Let $\Gamma^{\lambda}_{\mu\nu}$ be a given x-connexion and $\bar{x}$-scalar, and L^{p}_{qr} a given x-scalar and $\bar{x}$-connexion. Then the double-tensors defined by

$$T^{\mu(p)}_{\nu(q),\lambda} = \frac{\partial T^{\mu(p)}_{\nu(q)}}{\partial x^{\lambda}} + \Gamma^{\mu}_{\alpha\lambda} T^{\alpha(p)}_{\nu(q)} - \Gamma^{\alpha}_{\nu\lambda} T^{\mu(p)}_{\alpha(q)} \tag{8}$$

and

$$T^{\mu(p)}_{\nu(q);(r)} = \frac{\partial T^{\mu(p)}_{\nu(q)}}{\partial \bar{x}^{r}} + L^{p}_{ar} T^{\mu(a)}_{\nu(q)} - L^{a}_{qr} T^{\mu(p)}_{\nu(a)} \tag{9}$$

will be called respectively *the covariant x-derivative of $T^{\mu(p)}_{\nu(q)}$ with respect to the scalar-connexion $\Gamma^{\lambda}_{\mu\nu}$*, and *the covariant $\bar{x}$-derivative of $T^{\mu(p)}_{\nu(q)}$ with respect to the scalar-connexion L^{p}_{qr}*. That $T^{\mu(p)}_{\nu(q),\lambda}$ and $T^{\mu(p)}_{\nu(q);(r)}$ are in fact double-tensors of the types indicated by their suffixes follows easily from the definitions of the preceding paragraphs.

It will be observed that the definitions (8) and (9) are exactly similar to those of covariant derivatives in the ordinary tensor theory, so that there will be no difficulty in writing down the formulae for double-tensors of any rank. It may, however, be remarked that one of the covariant derivatives of a scalar-vector or scalar-tensor is the set of ordinary partial derivatives. Thus, for example, the covariant derivatives of the scalar-vector T^{μ} are given by the formulae

$$T^{\mu}{}_{,\lambda} = \frac{\partial T^{\mu}}{\partial x^{\lambda}} + \Gamma^{\mu}_{\alpha\lambda} T^{\alpha},$$

$$T^{\mu}{}_{;(r)} = \frac{\partial T^{\mu}}{\partial \bar{x}^{r}}.$$

Second-order covariant derivatives of double-tensors.

In order to avoid the introduction of unnecessarily complicated formulae we now consider simply a contravariant double-vector $T^{\mu(p)}$. The results obtained can easily be generalized to more complicated cases.

By definition,

$$T^{\mu(p)}{}_{,\lambda} = \frac{\partial T^{\mu(p)}}{\partial x^\lambda} + \Gamma^\mu_{\alpha\lambda} T^{\alpha(p)},$$

$$T^{\mu(p)}{}_{,\lambda,\sigma} = \frac{\partial T^{\mu(p)}{}_{,\lambda}}{\partial x^\sigma} + \Gamma^\mu_{\beta\sigma} T^{\beta(p)}{}_{,\lambda} - \Gamma^\beta_{\lambda\sigma}\, T^{\mu(p)}{}_{,\beta}$$

$$= \frac{\partial^2 T^{\mu(p)}}{\partial x^\sigma \partial x^\lambda} + \Gamma^\mu_{\alpha\lambda} \frac{\partial T^{\alpha(p)}}{\partial x^\sigma} + \frac{\partial \Gamma^\mu_{\alpha\lambda}}{\partial x^\sigma} T^{\alpha(p)} + \Gamma^\mu_{\beta\sigma} \frac{\partial T^{\beta(p)}}{\partial x^\lambda} +$$

$$+ \Gamma^\mu_{\beta\sigma}\Gamma^\beta_{\alpha\lambda} T^{\alpha(p)} - \Gamma^\beta_{\lambda\sigma} \frac{\partial T^{\mu(p)}}{\partial x^\beta} - \Gamma^\beta_{\lambda\sigma}\Gamma^\mu_{\alpha\beta} T^{\alpha(p)}.$$

Interchanging λ, σ and subtracting,* we have

$$T^{\mu(p)}{}_{,\lambda,\sigma} - T^{\mu(p)}{}_{,\sigma,\lambda} = P^\mu_{\alpha\lambda\sigma} T^{\alpha(p)} - 2\,\Sigma^\beta_{\lambda\sigma}\, T^{\mu(p)}{}_{,\beta} \tag{10}$$

where
$$P^\mu_{\alpha\lambda\sigma} = \frac{\partial \Gamma^\mu_{\alpha\lambda}}{\partial x^\sigma} - \frac{\partial \Gamma^\mu_{\alpha\sigma}}{\partial x^\lambda} + \Gamma^\mu_{\beta\sigma}\Gamma^\beta_{\alpha\lambda} - \Gamma^\mu_{\beta\lambda}\Gamma^\beta_{\alpha\sigma} \tag{11}$$

and
$$\Sigma^\beta_{\lambda\sigma} = \tfrac{1}{2}(\Gamma^\beta_{\lambda\sigma} - \Gamma^\beta_{\sigma\lambda}). \tag{12}$$

$P^\mu_{\alpha\lambda\sigma}$ and $\Sigma^\beta_{\lambda\sigma}$ are x-tensors as in the ordinary tensor-theory.† Since $\Gamma^\lambda_{\mu\nu}$ is an $\bar{x}$-scalar, they are also $\bar{x}$-scalars. $P^\mu_{\alpha\lambda\sigma}$ may be called *the scalar-tensor of curvature formed from* $\Gamma^\lambda_{\mu\nu}$, and $\Sigma^\beta_{\lambda\sigma}$ *the scalar-tensor of torsion formed from* $\Gamma^\lambda_{\mu\nu}$.

Similarly

$$T^{\mu(p)}{}_{;(q);(r)} - T^{\mu(p)}{}_{;(r);(q)} = R^p_{aqr} T^{\mu(a)} - 2 S^a_{qr} T^{\mu(p)}{}_{;(a)} \tag{13}$$

where
$$R^p_{aqr} = \frac{\partial L^p_{aq}}{\partial \bar{x}^r} - \frac{\partial L^p_{ar}}{\partial \bar{x}^q} + L^p_{br} L^b_{aq} - L^p_{bq} L^b_{ar} \tag{14}$$

and
$$S^a_{qr} = \tfrac{1}{2}(L^a_{qr} - L^a_{rq}) \tag{15}$$

* It is assumed that the components of $T^{\mu(p)}$ are continuous functions of both sets of variables for the ranges of values considered, so that

$$\partial^2 T^{\mu(p)}/\partial x^\sigma \partial x^\lambda = \partial^2 T^{\mu(p)}/\partial x^\lambda \partial x^\sigma.$$

† See, for example, Eisenhart, *Non-Riemannian Geometry* (American Math. Soc. Colloquium Publications, vol. viii, 1927), p. 7, (4.2).

are scalar-tensors which may be called respectively *the scalar-tensors of curvature and torsion formed from* L^p_{qr}.

These formulae, which of course are exactly similar to those of the ordinary tensor theory, give the permutation rules for successive covariant x-derivatives and $\bar{x}$-derivatives.

Consider now the covariant $\bar{x}$-derivative of $T^{\mu(p)}{}_{,\lambda}$. We have

$$\begin{aligned} T^{\mu(p)}{}_{,\lambda;(q)} &= \frac{\partial T^{\mu(p)}{}_{,\lambda}}{\partial \bar{x}^q} + L^p_{aq} T^{\mu(a)}{}_{,\lambda} \\ &= \frac{\partial^2 T^{\mu(p)}}{\partial \bar{x}^q \partial x^\lambda} + \Gamma^\mu_{\alpha\lambda} \frac{\partial T^{\alpha(p)}}{\partial \bar{x}^q} + L^p_{aq} \frac{\partial T^{\mu(a)}}{\partial x^\lambda} + \\ &\qquad + \Gamma^\mu_{\alpha\lambda} L^p_{aq} T^{\alpha(a)} + T^{\alpha(p)} \frac{\partial \Gamma^\mu_{\alpha\lambda}}{\partial \bar{x}^q}, \end{aligned} \tag{16}$$

by definition.

Similarly,

$$\begin{aligned} T^{\mu(p)}{}_{;(q),\lambda} &= \frac{\partial^2 T^{\mu(p)}}{\partial x^\lambda \partial \bar{x}^q} + L^p_{aq} \frac{\partial T^{\mu(a)}}{\partial x^\lambda} + \Gamma^\mu_{\alpha\lambda} \frac{\partial T^{\alpha(p)}}{\partial \bar{x}^q} + \\ &\qquad + L^p_{aq} \Gamma^\mu_{\alpha\lambda} T^{\alpha(a)} + T^{\mu(a)} \frac{\partial L^p_{aq}}{\partial x^\lambda}. \end{aligned} \tag{17}$$

Subtracting,

$$T^{\mu(p)}{}_{,\lambda;(q)} - T^{\mu(p)}{}_{;(q),\lambda} = \frac{\partial \Gamma^\mu_{\alpha\lambda}}{\partial \bar{x}^q} T^{\alpha(p)} - \frac{\partial L^p_{aq}}{\partial x^\lambda} T^{\mu(a)}. \tag{18}$$

Hence, *covariant x-derivatives are not in general permutable with covariant $\bar{x}$-derivatives.* In the particular case when $\Gamma^\mu_{\alpha\lambda}$ is independent* of the $(\bar{x}^i)$ and L^p_{aq} of the (x^μ), the right-hand side is zero and the covariant derivatives are permutable.

5. Geometrical considerations

The present paragraph contains remarks of a general character concerning the geometrical aspect of the analysis outlined above, and should give some indication of the kind of ideas to which the present theory leads.

Let $g_{\mu\nu}$ and $\bar{g}_{(pq)}$ be two distinct symmetrical† scalar-tensors of the types indicated by the suffixes. Let $(g^{\mu\nu})$, $(\bar{g}^{(pq)})$ be the matrices respectively reciprocal to $(g_{\mu\nu})$, $(\bar{g}_{(pq)})$. Then it is easily proved (as in the ordinary tensor theory) that the elements $g^{\mu\nu}$, $\bar{g}^{(pq)}$ are the components of symmetrical contravariant scalar-tensors.

* In a case such as this we shall say that the Γ's are x-connexions and $\bar{x}$-constants, and that the L's are $\bar{x}$-connexions and x-constants.

† i.e. such that $g_{\mu\nu} = g_{\nu\mu}$ and $\bar{g}_{(pq)} = \bar{g}_{(qp)}$.

Suppose now that (x^μ) and $(\bar{x}^i)$ are systems of coordinates in the Riemannian spaces whose metrics are respectively specified by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \tag{A}$$

$$d\bar{s}^2 = \bar{g}_{(pq)}d\bar{x}^p d\bar{x}^q, \tag{B}$$

the former therefore being of n dimensions and the latter of m.

Since $g_{\mu\nu}$ is an x-tensor and an $\bar{x}$-scalar, it follows in general that (A) will not determine a definite Riemannian space unless the $(\bar{x}^i)$ are kept constant; it is only when the $g_{\mu\nu}$ are actually independent of the $(\bar{x}^i)$ (i.e. are $\bar{x}$-constants) that $g_{\mu\nu}$ does not depend on the particular set of values possessed by the $(\bar{x}^i)$. Similarly, (B) will in general determine a definite Riemannian space only when the (x^μ) are kept constant. In other words, to each point of a Riemannian space (B) corresponds a Riemannian space (A), and vice versa. The formulae (A) and (B) thus really specify two multiply-infinite sets of Riemannian spaces.

Any double-tensor, $T^{\mu(p)}$ for example, will belong to both sets. In a space (A) it will be a set of m contravariant vectors $T^{\mu(1)}$, $T^{\mu(2)}$,..., $T^{\mu(m)}$, and in a space (B) a set of n contravariant vectors $T^{1(p)}$, $T^{2(p)}$,..., $T^{n(p)}$.

A special case may be of interest. Suppose that $n = m$ and that the $g_{\mu\nu}$ are $\bar{x}$-constants, so that (A) determines a single Riemannian space. Further, let the $\bar{g}_{(pq)}$ be the functions obtained from the $g_{\mu\nu}$ by replacing $x^1, x^2, \ldots, x^n$ respectively by $\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n$. Then (A) and (B) define the same space and (x^μ), $(\bar{x}^\nu)$ are the coordinates of two points of it, any double-tensor being a set of functions of the two sets of coordinates. For example, the length s of the arc of the geodesic joining the points (x^μ) and $(\bar{x}^\mu)$ is a double-scalar; the derivatives $\partial s/\partial x^\mu$, $\partial s/\partial \bar{x}^\nu$ are scalar-vectors; the derivative $\partial^2 s/\partial x^\mu \partial \bar{x}^\nu$ a double-vector, and so on.*

Return now to the general case when m and n are not necessarily equal and $g_{\mu\nu}$, $\bar{g}_{(pq)}$ are distinct scalar-tensors. The Christoffel symbols

$$\{\lambda\mu, \nu\} = \tfrac{1}{2}g^{\nu\alpha}\left\{\frac{\partial g_{\lambda\alpha}}{\partial x^\mu} + \frac{\partial g_{\mu\alpha}}{\partial x^\lambda} - \frac{\partial g_{\lambda\mu}}{\partial x^\alpha}\right\}, \tag{19}$$

$$\{\overline{pq, r}\} = \tfrac{1}{2}\bar{g}^{(ra)}\left\{\frac{\partial \bar{g}_{(pa)}}{\partial \bar{x}^q} + \frac{\partial \bar{g}_{(qa)}}{\partial \bar{x}^p} - \frac{\partial \bar{g}_{(pq)}}{\partial \bar{x}^a}\right\} \tag{20}$$

* Some properties of the function $\Omega = \frac{1}{2}s^2$ have been investigated in earlier papers. See, in particular, *Proc. London Math. Soc.*, 32 (1931), 87, in which derivatives of Ω with respect to both sets of variables are employed. It was the consideration of these derivatives which led to the ideas of the present paper.

are then scalar-connexions, and could be made the basis of a theory of covariant differentiation. Examples of another type of scalar-connexion are given below.

Double-ennuples.

As above, let

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{A'}$$

$$d\bar{s}^2 = \bar{g}_{(pq)}d\bar{x}^p d\bar{x}^q \tag{B'}$$

characterize two infinite sets of Riemannian spaces, $g_{\mu\nu}$ and $\bar{g}_{(pq)}$ being scalar-tensors.* For the purpose of this section it is necessary to suppose that $m = n$ (so that repeated Latin suffixes sum from 1 to n as well as Greek), but it is not now being assumed that $g_{\mu\nu}$ and $\bar{g}_{(pq)}$ are derivable from one another by an interchange of the x's and $\bar{x}$'s, though that may happen in particular cases.

Let $\lambda^{\mu(p)}$ be a contravariant double-vector belonging to these spaces, and write

$$\left.\begin{aligned} &\lambda^{(p)}_{\mu} \text{ for } g_{\mu\alpha}\lambda^{\alpha(p)} \\ &\lambda_{\mu(p)} \text{ for } g_{\mu\alpha}\bar{g}_{(p\alpha)}\lambda^{\alpha(\alpha)} \\ &\lambda^{\mu}_{(p)} \text{ for } \bar{g}_{(p\alpha)}\lambda^{\mu(\alpha)} \end{aligned}\right\}. \tag{21}$$

Let the double-vector be such that

$$\bar{g}_{(pq)}\lambda^{\mu(p)}\lambda^{\nu(q)} = g^{\mu\nu}, \tag{22}$$

which, in view of the assumed symmetry of $g_{\mu\nu}$, is equivalent to imposing $\frac{1}{2}n(n+1)$ conditions on the n^2 components $\lambda^{\mu(p)}$, a legitimate proceeding, since $n^2 > \frac{1}{2}n(n+1)$ if $n > 1$.

Multiplying (22) by $g_{\sigma\nu}$ and summing for ν, we get, in the notation of (21),

$$\lambda^{\mu}_{(q)}\lambda^{(q)}_{\sigma} = \delta^{\mu}_{\sigma}, \tag{23}$$

where δ^{μ}_{σ} is Kronecker's symbol. The matrices $(\lambda^{\mu}_{(q)})$, $(\lambda^{(q)}_{\sigma})$ are therefore reciprocal to one another, so it follows that

$$\lambda^{\mu}_{(q)}\lambda^{(p)}_{\mu} = \delta^{p}_{q}. \tag{24}$$

If both sides of this equation are multiplied by $\bar{g}^{(rq)}$ (and a summation performed for q), the resulting equation may be written

$$g_{\mu\nu}\lambda^{\mu(r)}\lambda^{\nu(p)} = \lambda^{(rp)}, \tag{25}$$

a formula analogous to (22).

If for the moment the $(\bar{x}^i)$ are kept fixed, it follows from (24) that each of the n x-vectors $\lambda^{\mu}_{(1)}, \lambda^{\mu}_{(2)}, \ldots, \lambda^{\mu}_{(n)}$ is orthogonal (in the sense of the ordinary tensor theory) to all but one of the n x-vectors

* $g_{\mu\nu}$ and $\bar{g}^{(pq)}$ may, in particular cases, be respectively $\bar{x}$-constants and x-constants, so that (A') and (B') may sometimes define unique spaces.

$\lambda^{\mu(1)}, \lambda^{\mu(2)}, \ldots, \lambda^{\mu(n)}$. Similarly, (23) shows that, the x's being constant, each of the $\bar{x}$-vectors $\lambda^{\mu}_{(q)}$ is orthogonal to all but one of the $\bar{x}$-vectors $\lambda_{\mu(q)}$.

A double-vector $\lambda^{\mu(p)}$ which satisfies equations (22), and hence also (25), will be said to constitute a *reciprocally orthogonal double-ennuple.**

Such a double-ennuple can be used to define what may be called the *scalar-tensors of rotation*, namely,

$$\bar{\gamma}_{(pqr)} = \lambda_{\mu(p),\,\nu}\lambda^{\mu}_{(q)}\lambda^{\nu}_{(r)} \tag{26}$$

and

$$\gamma_{\mu\nu\sigma} = \lambda_{\mu(p);\,(q)}\lambda^{(p)}_{\nu}\lambda^{(q)}_{\sigma}, \tag{27}$$

where

$$\lambda_{\mu(p),\nu} = \frac{\partial \lambda_{\mu(p)}}{\partial x^{\nu}} - \{\mu\nu,\,\alpha\}\lambda_{\alpha(p)}$$

and

$$\lambda_{\mu(p);(q)} = \frac{\partial \lambda_{\mu(p)}}{\partial \bar{x}^{q}} - \{\overline{pq,\,a}\}\lambda_{\mu(a)},$$

the Christoffel symbols being defined by (19) and (20). The name given to $\bar{\gamma}_{(pqr)}$ and $\gamma_{\mu\nu\sigma}$ is of course chosen on account of their similarity in Ricci's coefficients of rotation in the ordinary tensor theory,† to which indeed they reduce in certain particular cases. That they are actually scalar-tensors follows easily from the definitions.

It is worth noticing that, unlike Ricci's coefficients of rotation, $\bar{\gamma}_{(pqr)}$ and $\gamma_{\mu\nu\sigma}$ are *not* in general skew-symmetric in the first two suffixes. That is to say, the statements

$$\bar{\gamma}_{(pqr)} + \bar{\gamma}_{(qpr)} \neq 0 \tag{28}$$

and

$$\gamma_{\mu\nu\sigma} + \gamma_{\nu\mu\sigma} \neq 0 \tag{29}$$

are true except in special cases. For it follows easily from (24) that

$$\lambda_{\mu(p)}\lambda^{\mu}_{(q)} = \bar{g}_{(pq)},$$

and hence, by differentiating covariantly with respect to the x's and the Christoffel symbols $\{\lambda\mu,\,\nu\}$,

$$\lambda_{\mu(p),\nu}\lambda^{\mu}_{(q)} + \lambda_{\mu(p)}\lambda^{\mu}_{(q),\nu} = \bar{g}_{(pq),\nu},$$

where, of course, $\bar{g}_{(pq),\nu}$ is the first covariant x-derivative $\partial \bar{g}_{(pq)}/\partial x^{\nu}$ of

* It should be noticed that, the $\bar{x}$'s being fixed, the n x-vectors $\lambda^{\mu(1)}, \ldots, \lambda^{\mu(n)}$ do not in general constitute an orthogonal ennuple in the strict sense, for that would require $\lambda^{\mu(p)}\lambda^{(q)}_{\mu} = \delta^{p}_{q}$, which is not true. A similar remark is applicable, if the x's are kept fixed and $\lambda^{\mu(p)}$ is regarded as an $\bar{x}$-ennuple. It is for this reason that the word *reciprocal* is introduced into the definition.

† Levi-Civita, *Absolute Differential Calculus* (Blackie, 1927), p. 271.

the scalar-tensor $\bar{g}_{(pq)}$. Multiplying both sides of the last equation by $\lambda^{\nu}_{(r)}$ and summing for ν, we get

$$\bar{\gamma}_{(pqr)}+\bar{\gamma}_{(qpr)}=\bar{g}_{(pq),\nu}\lambda^{\nu}_{(r)}. \tag{30}$$

Similarly,

$$\gamma_{\mu\nu\sigma}+\gamma_{\nu\mu\sigma}=g_{\mu\nu\varsigma(q)}\lambda^{(q)}_{\sigma}, \tag{31}$$

so that the truth of (28) and (29) is established.

Lastly, it may be noted that the sets of functions defined by

$$\Delta^{\sigma}_{\mu\nu}=\lambda^{\sigma}_{(a)}\frac{\partial\lambda^{(a)}_{\mu}}{\partial x^{\nu}} \tag{32}$$

and

$$L^{p}_{qr}=\lambda^{(p)}_{\mu}\frac{\partial\lambda^{\mu}_{(q)}}{\partial\bar{x}^{r}} \tag{33}$$

constitute scalar-connexions of the types indicated by the suffixes. This is easily shown to be true by direct transformation of the x's and $\bar{x}$'s. For example,

$$'\Delta^{\sigma}_{\mu\nu}='\lambda^{\sigma}_{(a)}\frac{\partial'\lambda^{(a)}_{\mu}}{\partial x^{\nu}}=\lambda^{\sigma}_{(b)}\frac{\partial\bar{x}^{b}}{\partial\bar{x}'^{a}}\frac{\partial}{\partial x^{\nu}}\left\{\lambda^{(c)}_{\mu}\frac{\partial\bar{x}'^{a}}{\partial\bar{x}^{c}}\right\}=\lambda^{\sigma}_{(b)}\frac{\partial\bar{x}^{b}}{\partial\bar{x}'^{a}}\frac{\partial\bar{x}'^{a}}{\partial\bar{x}^{c}}\frac{\partial\lambda^{(c)}_{\mu}}{\partial x^{\nu}},$$

since the derivatives $\partial\bar{x}'^{a}/\partial\bar{x}^{c}$ are independent of the (x^{μ}). But

$$\frac{\partial\bar{x}^{b}}{\partial\bar{x}'^{a}}\frac{\partial\bar{x}'^{a}}{\partial\bar{x}^{c}}=\delta^{b}_{c},$$

whence it follows at once that

$$'\Delta^{\sigma}_{\mu\nu}=\Delta^{\sigma}_{\mu\nu},$$

and $\Delta^{\sigma}_{\mu\nu}$ is therefore an $\bar{x}$-scalar. By a similar method it can be shown that it is also an x-connexion,* and that L^{p}_{qr} is a scalar-connexion.

Other scalar-connexions of a similar character are those defined by

$$\Gamma^{\sigma}_{\mu\nu}=\lambda^{\sigma(a)}\frac{\partial\lambda_{\mu(a)}}{\partial x^{\nu}}, \tag{34}$$

and

$$H^{p}_{qr}=\lambda^{\mu(p)}\frac{\partial\lambda_{\mu(q)}}{\partial\bar{x}^{r}}. \tag{35}$$

Since

$$\frac{\partial\lambda_{\mu(a)}}{\partial x^{\nu}}=\frac{\partial}{\partial x^{\nu}}(\bar{g}_{(ab)}\lambda^{(b)}_{\mu})=\bar{g}_{(ab)}\frac{\partial\lambda^{(b)}_{\mu}}{\partial x^{\nu}}+\bar{g}_{(ab),\nu}\lambda^{(b)}_{\mu},$$

we get

$$\Gamma^{\sigma}_{\mu\nu}=\Delta^{\sigma}_{\mu\nu}+\bar{g}_{(ab),\nu}\lambda^{(b)}_{\mu}\lambda^{\sigma(a)}. \tag{36}$$

Similarly,

$$H^{p}_{qr}=L^{p}_{qr}+g_{\alpha\beta\gamma(r)}\lambda^{\beta}_{(q)}\lambda^{\alpha(p)}. \tag{37}$$

6. Conclusion

The remark of § 1 concerning the extension of the foregoing theory to tensor-functions of more than two sets of variables may now con-

* Cf. Eisenhart, op. cit., p. 48, (18.6).

veniently be referred to. A double-tensor requires two kinds of suffixes; a triple-tensor (a tensor-function of three sets of variables) would require three kinds, and so on. It is therefore obvious that a radical alteration in notation, if not in terminology, would be required if such a generalized theory were not to be submerged by the weight of its own symbolism.

In § 3 scalar-connexions are defined. It might be asked whether there would be any advantage in defining such concepts as those of 'double-connexions' (sets of functions behaving like affine-connexions for transformations of both sets of variables), of 'tensor-connexions', or of other hybrids of the same sort. The answer appears to be in the negative; for the covariant derivative of a double-tensor, formed with respect to a double-connexion (or a tensor-connexion) would not in general be a double-tensor.

The work contained in this paper has been presented as a generalization of the Absolute Differential Calculus of Ricci and Levi-Civita. From another point of view* it can be regarded as a *particular case* of the same Calculus. Suppose, for example, that $T_{\lambda(p)}$ is a covariant double-vector, the components being functions of the n variables (x^μ) and of the m variables $(\bar{x}^i)$. Write $\xi^1, \xi^2, \ldots, \xi^n$ respectively for $x^1, x^2, \ldots, x^n$, and $\xi^{n+1}, \xi^{n+2}, \ldots, \xi^{n+m}$ for $\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^m$; also $\xi'^1, \xi'^2, \ldots, \xi'^n$ for $x'^1, x'^2, \ldots, x'^n$ and $\xi'^{n+1}, \xi'^{n+2}, \ldots, \xi'^{n+m}$ for $\bar{x}'^1, \bar{x}'^2, \ldots, \bar{x}'^m$. Then the laws of transformation of $T_{\lambda(p)}$ give

$$'T'_{\lambda(p)} = T_{\alpha(a)} \frac{\partial \xi^\alpha}{\partial \xi'^\lambda} \frac{\partial \xi^a}{\partial \xi'^p}, \tag{38}$$

where it is understood that Greek indices sum from 1 to n and Latin from $n+1$ to $n+m$, and where $T_{\alpha(n+r)}$ is taken to mean what has hitherto been denoted by $T_{\alpha(r)}$.

Now if this restriction on the summations did not exist, and if $\xi^1, \xi^2, \ldots, \xi^{n+m}$ could be functions of *all* the ξ''s (instead of the first n of the ξ's being functions of the first n only of the ξ''s, and the last m of the ξ's functions of the last m only of the ξ''s), then (38) would give the ordinary law of transformation of the tensor $T_{\lambda(p)}$ of the second rank whose components are functions of the $n+m$ variables $\xi^1, \xi^2, \ldots, \xi^{n+m}$. So a double-vector $T_{\lambda(p)}$ can be regarded as an ordinary tensor of the second rank, specialized by the fact that the variables

* The remarks in the remainder of this paragraph are due to Professor T. Levi-Civita, to whom I owe my best thanks for his kind interest in this paper.

are divided into two separate groups of n and m respectively. A similar remark applies to any double-tensor.

This dividing of the variables into two groups is not unfamiliar, for it is commonly done when the four-dimensional space-time of Relativity is partitioned into 'space' *and* 'time'. Suspending now the summation convention, let

$$ds^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu}\, dx^\mu dx^\nu \tag{39}$$

specify the metric of space-time, the $g_{\mu\nu}$ being functions of the four coordinates x^0, x^1, x^2, x^3. Suppose that x^0 is known, intuitively or otherwise, to be identifiable with the time t, and that (x^1, x^2, x^3) are a given set of spatial coordinates. Then (39) may be written

$$ds^2 = V^2 dt^2 + 2dt \sum_{\mu=1}^{3} w_\mu\, dx^\mu - \sum_{\mu,\nu=1}^{3} a_{\mu\nu}\, dx^\mu dx^\nu, \tag{40}$$

where* $\qquad V^2 = g_{00},\ w_\mu = g_{0\mu},\ a_{\mu\nu} = -g_{\mu\nu} \qquad (\mu, \nu = 1, 2, 3).$

For transformations of the spatial coordinates (x^1, x^2, x^3) only, with t unaltered, V^2 behaves like a scalar, the w_μ like the three components of a covariant vector, and the $a_{\mu\nu}$ like the nine components of a symmetric covariant tensor of the second rank. And, if it were possible to attach a physical meaning to transformations of t only (x^1, x^2, x^3 remaining unaltered), V^2 would behave for such transformations like a covariant tensor of the second rank, having of course only one component since there is only one variable t; while w_1, w_2, w_3 would each behave like covariant vectors, and the $a_{\mu\nu}$ like scalars. So, having thus partitioned space-time into space *and* time, V^2 and $a_{\mu\nu}$ are scalar-tensors, and w_μ a double-vector according to the definitions of this paper, one of the two sets of variables being, in this special case, the single variable t.

Note added July 13: Since this paper was written there has appeared in the *Annals of Math.*, vol. xxxii (1931) 451 a paper by A. W. Tucker ('On generalised covariant differentiation') which anticipates to some extent the work presented above, although the point of view is different. Dr. Tucker has, in particular, devised a notation which renders practicable the consideration of tensor-functions of more than two sets of variables.

* The notation is borrowed from Levi-Civita, op. cit., p. 338.

NOTES ON THERMODYNAMICS

III. THE RELATION OF OSMOTIC PRESSURE TO CONCENTRATION FOR DILUTE SOLUTIONS

By E. A. MILNE

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It was shown by Lord Rayleigh,* and the proof has been generally followed in text-books,† that if a gas is dissolved in a solvent and the solution obeys Henry's Law, then the osmotic pressure of the solution obeys the laws of a perfect gas. More precisely, Lord Rayleigh showed that, if (*a*) the vapour-pressure of the solvent is negligible, and if also (*b*) the amount of gas dissolved at any pressure is proportional to the pressure, and if also (*c*) the volume of the solvent is unaltered by the solution in it of the gas, then the osmotic pressure of the dissolved substance is the same as the actual pressure the same amount of dissolved substance would exert if it were in the form of a perfect gas occupying the same volume as it occupies in the solution.

Lord Rayleigh's proof involved the consideration of a certain irreversible isothermal cycle, which may briefly be described as follows. Take a quantity of gas at a pressure p_0, volume v_0, expand it to a very large volume v_∞, pressure p_∞, bring this in contact with a volume V of the pure solvent, allow some of it to dissolve irreversibly (its pressure falling to p'_∞), compress it so that the rest of it goes into solution, and finally recover the pure solvent and the original gas by means of a semi-permeable membrane, doing work against the osmotic pressure. When certain adjustments are made (such as choosing an appropriate volume V) the cycle constitutes an irreversible one which approaches reversibility as $v_\infty \to \infty$, $p_\infty \to 0$. When the principle is applied that in this limit the net external work performed must be zero, the desired relation follows. This cycle gives great insight into the nature of irreversible processes by the intensive study of a particular example.

The object of the present note is to construct a proof of Lord Rayleigh's result, using the methods of Willard Gibbs. The final result is more general than Lord Rayleigh's. It is unnecessary to neglect the vapour-pressure of the solvent—it is merely necessary to

* *Nature*, 1897, p. 253; *Coll. Papers*, vol. vi, p. 267.

† e.g. Lewis, *Physical Chemistry*, vol. ii, p. 121 (1920), who quotes also van 't Hoff and Donnan.

suppose that the mixture of solvent-vapour and solute (gas) is an 'ideal gas-mixture' in the sense of Gibbs.* This is of course a very different matter from assuming that either gas or vapour separately are 'ideal gases'. Further, the final result appears in a form which is independent of the precise form of the gas-laws, whereas in Lord Rayleigh's proof Boyle's law is used. All cycles, whether reversible or irreversible, are of course avoided.

Consider the following system:

A	*B*	*C*
Pure solvent. (Pressure p.)	Solution. (Pressure p'.)	Gas-mixture. (Solvent vapour + solute (gas).) (Pressure p''.)	Pure solute (gas). (Pressure p'''.)
↑ Semi-permeable partition.	↑ Liquid/gas interface.	↑ Semi-permeable partition.	

Pure solvent, at pressure p, is in contact with solution, at pressure p', across a semi-permeable partition A. The solution is in contact with its own vapour, a mixture of solvent-vapour and solute (gas) across a liquid/gas interface B, the pressure of the gas-mixture being $p''(=p')$. The gas-mixture is in contact with pure solute (gas), at pressure p''', across a semi-permeable partition C.

Let the suffix 1 denote the material of which the solvent is made, whatever its phase; let the suffix 2 denote the material of which the solute is made, whatever its phase. The primes denote phases. Let μ with its appropriate suffix and prime denote a partial potential; let ρ with its appropriate suffix and prime denote a density, total density in a pure phase and partial density in a mixture. This gives a completely adequate notation. For example, μ_2'' denotes the potential of the solute in the gas-mixture, ρ_1 denotes the density of the pure solvent, ρ_2' the partial density of the solute in the solution.

The temperature is given to be the same everywhere, T.

We first inquire as to the number of degrees of freedom of the system. By Gibbs's condition of the equality of the partial potentials of a constituent through all phases in contact we have

$$\begin{aligned} \mu_1 = \mu_1' = \mu_1'' \\ \mu_2' = \mu_2'' = \mu_2'''. \end{aligned} \qquad (1)$$

* 'The Equilibrium of Heterogenous Substances': *Coll. Works*, 1, p. 155.

By the equality of pressure across the liquid/gas interface we have

$$p' = p''. \tag{2}$$

From the fundamental equations of the different phases we have

$$\begin{aligned} p &= f(\mu_1, T) \\ p' &= f'(\mu_1', \mu_2', T) \\ p'' &= f''(\mu_1'', \mu_2'', T) \\ p''' &= f'''(\mu_2''', T). \end{aligned} \tag{3}$$

We have thus 9 relations. Assuming T to be given, the number of unknowns is 10, namely, 6 partial potentials and 4 pressures. The system thus possesses one degree of freedom, which may conveniently be taken to be any one of the pressures.

Let us make use of this one degree of freedom to vary one of the pressures by a small amount; the other pressures have corresponding variations. By the theory of fundamental equations we have

$$dp = \rho_1 d\mu_1 \tag{4}$$

$$dp' = \rho_1' d\mu_1' + \rho_2' d\mu_2' \tag{5}$$

$$dp'' = \rho_1'' d\mu_1'' + \rho_2'' d\mu_2'' \tag{6}$$

$$dp''' = \rho_2''' d\mu_2'''. \tag{7}$$

By equations (1) the differentials $d\mu_1$, $d\mu_1'$, $d\mu_1''$ are all equal, and the differentials $d\mu_2'$, $d\mu_2''$, $d\mu_2'''$ are all equal.

The osmotic pressure P of the solution is given by

$$P = p' - p, \tag{8}$$

whence

$$dP = dp' - dp,$$

or by (4), (5), and (7), taking account of the equalities of the $d\mu_1$'s and of the $d\mu_2$'s,

$$dP = \left(\frac{\rho_1'}{\rho_1} - 1\right) dp + \frac{\rho_2'}{\rho_2'''}\, dp'''. \tag{9}$$

If now (a) the volume of the solvent at pressure p is equal to that of the same mass of solvent in the solution at pressure p', we have

$$\rho_1' = \rho_1.^* \tag{a}$$

If (b) the gas being dissolved obeys Henry's law, we have that the partial density ρ_2' of the solute in solution is proportional, at a given temperature, to the partial density ρ_2'' of the solute in the gas-mixture in contact with it, and so

$$\frac{\rho_2'}{\rho_2''} = \text{constant} = k. \tag{b}$$

* This is different from the usual enunciation of Henry's law.

If (*c*) the gas-mixture of solvent vapour and solute is an 'ideal gas-mixture', the potentials μ_2'' and μ_2''' being equal, we must have

$$\rho_2'' = \rho_2'''. \qquad (c)$$

The physical assumptions (*a*), (*b*), (*c*) introduced into the exact thermodynamic relation (9) give

$$dP = k dp'''. \qquad (10)$$

Integrating and noting that P and p''' must vanish together, we have

$$P = k p''',$$

whence, by (*b*) and (*c*),

$$\frac{P}{p'''} = \frac{\rho_2'}{\rho_2'''}$$

or

$$\frac{P}{\rho_2'} = \frac{p'''}{\rho_2'''}. \qquad (11)$$

Thus the ratio of osmotic pressure to partial density for the dissolved substance is equal to the ratio of gas-pressure to density for the same substance pure when in the form of gas at the same temperature. This result is independent of the form of the gas-laws.

In the case of a classical ideal gas we have

$$p'''/\rho_2''' = (R/m)T,$$

whence

$$P/\rho_2' = (R/m)T,$$

where R is the gas-constant and m the molecular weight. This latter particular result is the usual form of the theorem.

GENERALIZED MEANS AND THE SUMMABILITY OF FOURIER SERIES

By L. S. BOSANQUET AND E. H. LINFOOT

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1. Introduction. The series $\sum a_n$ is said to be summable by Riesz's arithmetic means of order $\alpha \geqslant 0$ or, what is equivalent, to be summable (C, α) to the sum s when

$$\sum_{n<\omega} \left(1-\frac{n}{\omega}\right)^{\alpha} a_n \to s \tag{1.01}$$

as $\omega \to \infty$. Writing

$$A(\omega) = \sum_{n<\omega} a_n, \tag{1.02}$$

we may express this by saying that as $\omega \to \infty$, $A(\omega) \to s$ (C, α).

In the same way, if the function $\phi(t)$ is integrable in $(0, 1)$ in the Lebesgue or Denjoy sense, $\phi(t)$ is said to tend (C, α) to the limit l as $t \to 0$, where $\alpha > 0$, when

$$\alpha \int_0^1 (1-u)^{\alpha-1} \phi(tu)\, du \to l \tag{1.03}$$

as $t \to 0$. This we may write

$$\phi(t) \to l \ (C, \alpha)$$

as $t \to 0$; when $\phi(0) = l$ we have continuity (C, α) or 'mean continuity of order α' at $t = 0$. If $\phi(t) \to l$ in the ordinary sense, it is said to 'tend $(C, 0)$ to l'.

The relations between the Cesàro summability of the Fourier series of a function $f(x)$ and the mean continuity at $t = 0$ of

$$\phi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)-2s\}$$

have been investigated by various writers,† and two of the main results are as follows:‡

THEOREM A. *If $f(t)$ is periodic with period 2π and integrable-L, and $\phi(t) = \frac{1}{2}\{f(x+t)+f(x-t)-2s\} \to 0$ (C, α) as $t \to 0$, where $\alpha \geqslant 0$, then the Fourier series of $f(t)$ is summable $(C, \alpha+\delta)$ to the sum s at $t = x$ for every $\delta > 0$.*

THEOREM B. *If $f(t)$ is periodic with period 2π and integrable-L, and if its Fourier series is summable (C, α) to the sum s at $t = x$, where $\alpha \geqslant 0$, then $\phi(t) = \frac{1}{2}\{f(x+t)+f(x-t)-2s\} \to 0$ $(C, \alpha+1+\delta)$ as $t \to 0$ for every $\delta > 0$.*

† Hardy and Littlewood (**4**), (**5**), (**6**), (**7**), Pollard (**12**), Paley (**11**), Verblunsky (**15**), Bosanquet (**1**). ‡ See e.g. Bosanquet (**1**) and Paley (**11**).

If $\frac{1}{2}a_0+\sum(a_m\cos mx+b_m\sin mx)$ is the Fourier series of $f(x)$, the series $\sum(b_m\cos mx-a_m\sin mx)$ is called its allied series. There are similar relations between the summability of the allied series and the continuity at $t=0$ of the function $\psi(t)=\frac{1}{2}\{f(x+t)-f(x-t)\}$, namely†

THEOREM C. *If $f(t)$ is periodic with period 2π and integrable-L, and $\psi(t)=\frac{1}{2}\{f(x+t)-f(x-t)\}\to 0$ (C,α) as $t\to 0$, where $\alpha\geqslant 0$, then the allied series of $f(t)$ is, for every $\delta>0$, summable $(C,\alpha+\delta)$ to the sum*

$$\lim_{t\to 0}\frac{2}{\pi}\int_t^\infty\frac{\psi(t)}{t}\,dt, \tag{1.04}$$

provided this limit exists‡ by some Cesàro mean.

THEOREM D. *If $f(t)$ is periodic with period 2π and integrable-L, and the allied series of $f(t)$ is summable (C,α) to s at $t=x$, where $\alpha\geqslant 0$, then $\psi(t)=\frac{1}{2}\{f(x+t)-f(x-t)\}\to 0$ $(C,\alpha+1+\delta)$ as $t\to 0$, for every $\delta>0$, and the limit*

$$\lim_{t\to 0}\frac{2}{\pi}\int_t^\infty\frac{\psi(u)}{u}\,du \tag{1.05}$$

exists by some Cesàro mean and has the value s.

It has been shown by considering special cases that all these theorems are false with $\delta=0$.§

The main object of this paper is to set up a double scale of summability which will enable us to obtain more precise information about the relations between mean continuity and summability. To do this we generalize the Cesàro-Riesz scale by introducing logarithmic factors.‖

DEFINITION 1.1. (*Generalized summability.*) $\sum a_n$ *is said to be summable (α,β) to the sum s, where either $\alpha>0$, or $\alpha=0$, $\beta\geqslant 0$, if as $\omega\to\infty$*

$$A^{(\alpha,\beta)}(\omega)=\sum_{n<\omega}B\left(1-\frac{n}{\omega}\right)^\alpha\log^{-\beta}\frac{C}{1-(n/\omega)}a_n\to s, \tag{1.11}$$

where $B=(\log C)^\beta$, for all sufficiently large values of C.

The reason for introducing the undetermined parameter C will

† Paley (**11**).

‡ The integrals in (1.04), (1.05) are in general of the Cauchy-Lebesgue type.

§ See Paley (**5**).

‖ An analogous scale has been given by Zygmund (**18**), who obtains results concerning power series. But his definition is not equivalent to ours. See Bosanquet and Linfoot (**2**), p. 118, footnote.

appear later.† More generally, we may say that $\sum a_n$ is summable $(\alpha_0, \alpha_1, \ldots, \alpha_k)$ to s, where the first non-zero α is positive, if as $\omega \to \infty$

$$A^{(\alpha_0, \alpha_1, \ldots, \alpha_k)}(\omega) = \sum_{n<\omega} B\left(1-\frac{n}{\omega}\right)^{\alpha_0} \prod_{\nu=1}^{k} \log_\nu^{-\alpha_\nu} \frac{C_\nu}{1-(n/\omega)} a_n \to s \quad (1.12)$$

for all sufficiently large $C_1, C_2, \ldots, C_k$, where $B = \prod_{\nu=1}^{k} \log_\nu^{\alpha_\nu} C_\nu$.

All these definitions may be written in the form

$$\lim_{\omega\to\infty} \sum_{n\leqslant\omega} \Phi\left(1-\frac{n}{\omega}\right) a_n = s, \quad (1.13)$$

where $\Phi(u)$ is a suitable summing function and $\Phi(0) = 0$, $\Phi(1) = 1$. Defining $A(\omega)$ by (1.02), we have

$$\sum_{n\leqslant\omega} \Phi\left(1-\frac{n}{\omega}\right) a_n = \sum_{n\leqslant\omega} \Phi\left(1-\frac{n}{\omega}\right)\{A(n)-A(n-1)\}$$

$$= \sum_{n=0}^{[\omega]-1} \left\{\Phi\left(1-\frac{n}{\omega}\right)-\Phi\left(1-\frac{n+1}{\omega}\right)\right\} A(n) + \left\{\Phi\left(1-\frac{[\omega]}{\omega}\right)-\Phi(0)\right\} A[\omega]$$

$$= \sum_{n=0}^{[\omega]-1} \frac{1}{\omega} A(n) \int_n^{n+1} \Phi'\left(1-\frac{u}{\omega}\right) du + \frac{1}{\omega} A([\omega]) \int_{[\omega]}^{\omega} \Phi'\left(1-\frac{u}{\omega}\right) du$$

$$= \frac{1}{\omega} \int_0^{\omega} \Phi'\left(1-\frac{u}{\omega}\right) A(u)\, du = \int_0^1 \Phi'(1-v) A(\omega v)\, dv.$$

The generalized sum is therefore simply

$$\lim_{\omega\to\infty} \int_0^1 \Phi'(1-u) A(\omega u)\, du. \quad (1.14)$$

DEFINITION 1.2. (*Generalized mean continuity.*) *We say that*

$$f(t) \to l \qquad (\alpha, \beta) \quad (1.21)$$

as $t \to 0$, *where either* $\alpha > 0$, *or* $\alpha = 0$, $\beta > 0$, *if as* $t \to 0$

$$M^{(\alpha,\beta)} f(t) = \int_0^1 \Phi_{\alpha,\beta}(1-v) f(tv)\, dv \to l \quad (1.22)$$

for all sufficiently large C,‡ *where*

$$\Phi_{\alpha,\beta}(u) = Bu^\alpha \log^{-\beta}(C/u), \qquad B = (\log C)^\beta.$$

When $f(0) = l$, we have 'continuity (α, β)' at $t = 0$. 'Continuity $(0, 0)$' is defined as ordinary continuity. Here and throughout, the setting

† For instance, in the proofs of Theorems 3.3, 3.4. We show, however, in Theorem 3.2 that it is equivalent to say 'for every $C > 1$'.

‡ Or for every $C > 1$. See Theorem 3.2.

down of an integral is to be taken as asserting its existence in the Lebesgue sense unless the contrary is stated. Thus the integral in (1.22) is supposed to exist for all sufficiently small values of t.

2. Before proving consistency theorems for these definitions we establish the following fundamental lemma.

LEMMA 2.1. *If $K(u) > 0$ is differentiable and $uK'(u)/K(u)$ increasing (i.e. non-decreasing) in the interval $0 \leqslant u < 1$ and $K(0) \neq 0$, and if $f(x)$ is integrable-L in every finite interval $(0, X)$, then*

$$\left| x \int_0^\eta K(u) f(xu)\, du \right| \leqslant \overline{\operatorname{bound}}_{0 \leqslant t \leqslant x} \left| t \int_0^1 K(u) f(tu)\, du \right| \tag{2.11}$$

for $0 \leqslant \eta \leqslant 1$, provided the integral on the right exists for $0 \leqslant t \leqslant x$.

The hypothesis '$uK'(u)/K(u)$ increasing' is equivalent to '$\log K(u)$ is a convex function of $\log u$'. In particular this is satisfied if $\log K(u)$ is an increasing convex function of u.

This is really an extension of a result due to M. Riesz,† which itself includes the particular case of Lemma 2.1, where $K(u) = \alpha(1-u)^{\alpha-1}$, $(0 < \alpha < 1)$ and $f(x)$ is a step-function. Hardy and Littlewood extended it to the case where $f(x)$ is integrable-L,‡ and our argument is based on theirs.

Proof of Lemma 2.1. We may suppose

$$\int_0^1 K(u)\, du = 1 \text{ and } \limsup_{t \to +0} \left| t \int_0^1 K(u) f(tu)\, du \right| < \infty$$

without loss of generality. Let $\delta > 0$ and

$$B(x) = \overline{\operatorname{bound}}_{0 \leqslant t \leqslant x} \left| t \int_0^1 K(u) f(tu)\, du \right| + \delta. \tag{2.11}$$

Then it will be sufficient to show that

$$\left| x \int_0^\eta K(u) f(xu)\, du \right| < B(x)$$

for $\delta > 0$, $x > 0$, $0 < \eta \leqslant 1$, provided the integrals exist.

† Riesz (**13**). See also Hardy and Riesz (**8**), p. 28.

‡ In an unpublished investigation. Riesz has also employed a more general form. If $0 < \alpha < 1$ and $0 \leqslant \xi < x$, then there is a τ, $0 \leqslant \tau < \xi$, such that

$$g(x, \xi) = \frac{1}{\Gamma(\alpha)} \int_0^\xi \phi(t)(x-t)^{\alpha-1}\, dt = \frac{1}{\Gamma(\alpha)} \int_0^\tau \phi(t)(\tau-t)^{\alpha-1}\, dt = \phi_\alpha(\tau),$$

provided only $\phi_\alpha(t)$ is continuous for $0 \leqslant t < \xi$. See Riesz (**14**), Verblunsky (**16**).

(i) *Suppose $f(x)$ continuous in $x \geqslant 0$.* Let

$$N(\xi) = \overline{\underset{0<\eta\leqslant 1,\, 0\leqslant x<\xi}{\text{bound}}} \left| \frac{x}{B(x)} \int_0^\eta K(u) f(xu)\, du \right|.$$

Then $N(\xi)$ is an increasing function; let λ be the upper bound of the numbers ξ for which $N(\xi) < 1$. We shall prove that $\lambda = +\infty$, which will establish the first part.

Suppose λ finite. Then $\lambda > 0$, for there is an open interval $(0, \zeta)$ throughout which

$$\frac{B(x)}{x} \geqslant \frac{\delta}{x} > \max_{0\leqslant t\leqslant x} |f(t)|,$$

and hence

$$\left| \frac{x}{B(x)} \int_0^\eta K(u) f(xu)\, du \right| \leqslant \frac{x}{B(x)} \max_{0\leqslant t\leqslant x} |f(t)| \int_0^1 K(u)\, du < 1.$$

Let $\mu > \lambda$. Then we can find μ', η' such that $\lambda < \mu' < \mu$, $0 < \eta' \leqslant 1$ and

$$\left| \frac{\mu'}{B(\mu')} \int_0^{\eta'} K(u) f(\mu' u)\, du \right| \geqslant 1,$$

i.e. on writing $u = \eta' v$ and $\mu'' = \mu' \eta'$,

$$\left| \frac{\mu''}{B(\mu')} \int_0^1 K(\eta' v) f(\mu'' v)\, dv \right| \geqslant 1$$

or

$$\frac{B(\mu'')}{B(\mu')} \left| \frac{\mu''}{B(\mu'')} \int_0^1 \frac{K(\eta' v)}{K(v)} K(v) f(\mu'' v)\, dv \right| \geqslant 1. \tag{2.12}$$

Now, since $B(x)$ is an increasing function, $B(\mu'') \leqslant B(\mu')$ and, since $uK'(u)/K(u)$ is an increasing function, $K'(v)/K(v) \geqslant \eta' K'(\eta' v)/K(\eta' v)$. Thus

$$\frac{d}{dv} \frac{K(\eta' v)}{K(v)} = \frac{\eta' K'(\eta' v) K(v) - K(\eta' v) K'(v)}{(K(v))^2} \leqslant 0$$

and $K(\eta' v)/K(v)$ is a decreasing function of v in the interval $(0, 1)$, with the value 1 for $v = 0$. Applying the second mean-value theorem to (2.12), it follows that there is an η'' such that $0 < \eta'' \leqslant 1$ and

$$\left| \frac{\mu''}{B(\mu'')} \int_0^{\eta''} K(v) f(\mu'' v)\, dv \right| \geqslant 1.$$

Writing $\mu^{(3)} = \mu''\eta''$ and proceeding in the same way, we see that there is an $\eta^{(3)}$ such that $0 < \eta^{(3)} \leqslant 1$ and

$$\left| \frac{\mu^{(3)}}{B(\mu^{(3)})} \int_0^{\eta^{(3)}} K(u) f(\mu^{(3)} u)\, du \right| \geqslant 1;$$

and generally

$$\left| \frac{\mu^{(n)}}{B(\mu^{(n)})} \int_0^{\eta^{(n)}} K(u) f(\mu^{(n)} u)\, du \right| \geqslant 1$$

for $n = 1, 2, 3, \ldots$, where $0 < \eta^{(n)} \leqslant 1$ and

$$\mu > \mu' \geqslant \mu'' \geqslant \ldots \geqslant \mu^{(n)} \geqslant \lambda > 0.$$

As $n \to \infty$, $\mu^{(n)} \to \mu^*$, where $\mu^* \geqslant \lambda > 0$, and $\eta^{(n)} = \mu^{(n+1)}/\mu^{(n)} \to 1$, so that

$$\frac{\mu^{(n)}}{B(\mu^{(n)})} \int_0^{\eta^{(n)}} K(u) f(\mu^{(n)} u)\, du \to \frac{\mu^*}{B(\mu^*)} \int_0^1 K(u) f(\mu^* u)\, du.$$

For $\int_0^1 K(u) f(\mu^{(n)} u)\, du \to \int_0^1 K(u) f(\mu^* u)\, du$, since $f(x)$ is continuous; $\mu^{(n)}/B(\mu^{(n)}) \to \mu^*/B(\mu^*)$, since $B(x)$ is continuous; and

$$\int_{\eta^{(n)}}^1 K(u) f(\mu^{(n)} u)\, du \to 0,$$

since $f(x)$ is bounded in $(0, \mu)$. It follows that if $\lambda < +\infty$,

$$\left| \frac{\mu^*}{B(\mu^*)} \int_0^1 K(u) f(\mu^* u)\, du \right| \geqslant 1.$$

But by (2.11)

$$\left| \mu^* \int_0^1 K(u) f(\mu^* u)\, du \right| < B(\mu^*).$$

Thus $\lambda = +\infty$ and the lemma is proved for continuous $f(x)$.

(ii) *Suppose $f(x)$ is integrable-L*; and assume for simplicity that $f(x) = 0$ for $x < 0$. Write

$$ {}_n\phi(x) = n \int_{x-1/n}^{x} \phi(u)\, du. \tag{2.13}$$

Then ${}_nf(x) \to f(x)$ for almost all x, since $f(x)$ is integrable-L, and

$$|{}_n\phi(u)| \leqslant \max_{x-1/n \leqslant t \leqslant x} |\phi(t)|. \tag{2.14}$$

Now writing

$$M_n f(x) = x\int_0^1 K(u)\,{}_n f(xu)\,du = \int_0^x K(1-v/x)\,{}_n f(x-v)\,dv$$

$$= n\int_0^x K(1-v/x)\int_{x-1/n}^x f(w-v)\,dw\,dv$$

$$= n\int_{x-1/n}^x dw\int_0^x K(1-v/x)f(w-v)\,dv$$

$$= n\int_{x-1/n}^x dw\int_0^w K(1-v/x)f(w-v)\,dv = {}_n[Mf(x)],$$

then, since $f(w-v)=0$ for $v>w$, we have, by (2.14),

$$\max_{0\leqslant t\leqslant x}|M_n f(x)| = \max_{0\leqslant t\leqslant x}|{}_n[Mf(t)]| \leqslant \max_{0\leqslant t\leqslant x}|Mf(t)|$$

$$= \max_{0\leqslant t\leqslant x}\left|t\int_0^1 K(w)f(tu)\,du\right|.$$

But, since ${}_n f(x)$ is continuous,

$$\left|x\int_0^\eta K(u)\,{}_n f(xu)\,du\right| \leqslant \max|M\,{}_n f(t)|$$

by the first part; hence the lemma will be established if we prove that as $n\to\infty$

$$x\int_0^\eta K(u)\,{}_n f(xu)\,du \to x\int_0^\eta K(u)f(xu)\,du.$$

But, $x\geqslant 0$ being fixed,

$${}_n f_1(\zeta) = \int_0^\zeta {}_n f(u)\,du = n\int_0^\zeta dt\int_{t-1/n}^t f(w)\,dw = n\int_0^\zeta \{f_1(t)-f_1(t-1/n)\}\,dt$$

$$= n\{f_2(\zeta)-f_2(\zeta-1/n)\}\to f_1(\zeta)$$

uniformly for $0\leqslant\zeta\leqslant x$, since $f_1(\zeta)$ is continuous, where

$$f_1(x) = \int_0^x f(t)\,dt, \qquad f_2(x) = \int_0^x f_1(t)\,dt.$$

Hence finally

$$x\int_0^\eta K(u)\,{}_n f(xu)\,du = x[K(u)\,{}_n f_1(xu)]_0^\eta - \int_0^\eta K'(u)\,{}_n f_1(xu)\,du$$

$$\to [K(u)f_1(xu)]_0^\eta - \int_0^\eta K'(u)f_1(xu)\,du = x\int_0^\eta K(u)f(xu)\,du.$$

From this lemma we easily deduce three more special results.

LEMMA 2.2. *If*

(i) $K(u)$, $f(x)$ *satisfy the conditions of Lemma* 2.1,

(ii) $\limsup_{t\to+0}\left|t\int_0^1 K(u)f(tu)\,du\right|<\infty$,

(iii) $\int_0^1 K(u)f(xu)\,du=o(x^\lambda)$ *either as* $x\to+0$ *or as* $x\to\infty$, *where* $\lambda>-1$, *then*

$$\int_0^\eta K(u)f(xu)\,du=o(x^\lambda), \tag{2.21}$$

uniformly for $0\leqslant\eta\leqslant1$, *as* $x\to+0$ *or as* $x\to\infty$ *respectively.*

Proof. We have

$$\overline{\text{bound}}_{0\leqslant t\leqslant x}\left|t\int_0^1 K(u)f(tu)\,du\right|<\epsilon x^{1+\lambda},$$

provided $x<x_0(\epsilon)$ or $x>X_0(\epsilon)$ respectively, by Lemma 2.1. It follows that, under the same proviso,

$$\left|x\int_0^\eta K(u)f(xu)\,du\right|<\epsilon x^{1+\lambda}$$

for $0\leqslant\eta\leqslant1$.

LEMMA 2.3. *If*

(i) $K(u)$, $f(x)$ *satisfy the conditions of Lemma* 2.1,

(ii) $\limsup_{t\to+0}\left|t\int_0^1 K(u)f(tu)\,du\right|<\infty$,

(iii) $k(u)$ *is of bounded variation in* $(0,1)$,

(iv) $\int_0^1 K(u)f(xu)=o(x^\lambda)$ *either as* $x\to+0$ *or as* $x\to\infty$, *where* $\lambda>-1$, *then*

$$\int_0^\eta K(u)k(u)f(xu)\,du=o(x^\lambda), \tag{2.31}$$

uniformly for $0\leqslant\eta\leqslant1$, *as* $x\to+0$ *or* $x\to\infty$ *respectively.*

Proof. We can write $k(u)=k_1(u)-k_2(u)$, where $k_1(u)$ and $k_2(u)$ are positive decreasing in $(0,1)$. Then

$$\int_0^\eta K(u)k(u)f(xu)\,du=k_1(0)\int_0^{\eta_1}K(u)f(xu)\,du-k_2(0)\int_0^{\eta_2}K(u)f(xu)\,du,$$

where $0<\eta_1,\eta_2\leqslant1$, and by Lemma 2.2 this is $o(x^\lambda)$ as $x\to+0$ or $x\to\infty$ respectively.

LEMMA 2.4. *If*

(i) $K(u)$ *and* $K(u)k(u)$ *both satisfy the conditions imposed on* K *in Lemma* 2.1 *and* $f(x)$ *satisfies those imposed on* f,

(ii) $k(u)$ *and* $1/k(u)$ *are both of bounded variation in* $(0, 1)$,

(iii)
$$\limsup_{t\to+0}\left|t\int_0^1 K(u)f(tu)\,du\right|<\infty,$$
$$\limsup_{t\to+0}\left|t\int_0^1 K(u)k(u)f(tu)\,du\right|<\infty,$$

then the statements

$$\int_0^1 K(u)f(xu)\,du=o(x^\lambda)\ \text{as}\ x\to+0,+\infty$$

and

$$\int_0^1 K(u)k(u)f(xu)\,du=o(x^\lambda)\ \text{as}\ x\to+0,+\infty$$

are equivalent for $\lambda>-1$.

Proof. By Lemma 2.2.

3. *Equivalent definitions. Consistency theorems.*
We could alternatively define the (α, β)-sum of a series $\sum a_n$ as

$$\left.\begin{aligned}&\lim_{\omega\to\infty}\int_0^1 G(1-u)^{\alpha-1}\log^{-\beta}\frac{C}{1-u}A(\omega u)\,du=s \qquad (\alpha>0),\\ &\lim_{\omega\to\infty}\int_0^1 G(1-u)^{-1}\log^{-\beta-1}\frac{C}{1-u}A(\omega u)\,du=s \qquad (\alpha=0,\beta>0),\end{aligned}\right\}\tag{3.01}$$

where, as before, $A(x)$ is defined by (1.02), provided the limit exists and has the same value for all sufficiently large C. The constant $G=G(\alpha,\beta,C)$ is chosen so that, when $A(x)=1$ for $x\geqslant 0$, the limit s is 1.

In the same way we could say that $f(t)$ tends (α, β) to the limit l if, for all sufficiently large C,

$$\left.\begin{aligned}&\lim_{t\to0}\int_0^1 G(1-u)^{\alpha-1}\log^{-\beta}\frac{C}{1-u}f(tu)\,du=l \qquad (\alpha>0),\\ &\lim_{t\to0}\int_0^1 G(1-u)^{-1}\log^{-\beta-1}\frac{C}{1-u}f(tu)\,du=l \qquad (\alpha=0,\beta>0).\end{aligned}\right\}\tag{3.02}$$

To justify this we establish the following theorem:

THEOREM 3.1. *The new definitions of* (α, β)*-summability and continuity are equivalent to the old.*

Proof. We may suppose without loss of generality that $s = 0$, $l = 0$, and that $t > 0$ in (3.02), so that we need only consider right-handed limits.

Let the integer $h \geqslant 0$ be defined as follows:

$$\left.\begin{array}{lllll} \text{either} & h < \alpha < h+1 & \beta > = < 0 & \\ \text{or} & \alpha = h+1 & \beta \leqslant 0 & \end{array}\right\} \text{(case i)}, \qquad \text{or} \quad h = \alpha \quad \beta > 0 \quad \text{(case ii)}. \tag{3.11}$$

If we take

$$\begin{aligned} \Phi_{\alpha,\beta}(u) &= Bu^{\alpha}\log^{-\beta}(C/u) && (\alpha > -1) \\ &= Bu^{-1}\log^{-\beta-1}(C/u) && (\alpha = -1,\ \beta > 0), \end{aligned} \tag{3.12}$$

where B is chosen so that $\Phi_{\alpha,\beta}(1) = 1$, the (α, β)-sum $A^{(\alpha,\beta)}(\omega)$ can be written

$$\int_0^1 \Phi'_{\alpha,\beta}(1-u)A(\omega u)\,du$$

by (1.14). If we integrate h times by parts and write

$$A_m(x) = \int_0^x A_{m-1}(t)\,dt, \qquad A_0(t) = A(t),$$

this becomes

$$\omega^{-h}\int_0^1 \Phi^{(h+1)}_{\alpha,\beta}(1-u)A_h(\omega u)\,du,$$

while the integrals in (3.01), namely

$$GB^{-1}\int_0^1 \Phi_{\alpha-1,\beta}(1-u)A(\omega u)\,du,$$

can be written

$$GB^{-1}\omega^{-h}\int_0^1 \Phi^{(h)}_{\alpha-1,\beta}(1-u)A_h(\omega u)\,du$$

in the same way. Again the integral of (1.22) can be written

$$t^{-h}\int_0^1 \Phi^{(h+1)}_{\alpha,\beta}(1-v)f_h(tv)\,dv,$$

while those of (3.02) become

$$t^{-h}\int_0^1 \Phi^{(h)}_{\alpha-1,\beta}(1-v)f_h(tv)\,dv.$$

What we have to prove is thus the equivalence of

$$\int_0^1 \Phi^{(h)}_{\alpha-1,\beta}(1-u)f(xu)\,du = o(x^h) \tag{3.13}$$

and

$$\int_0^1 \Phi^{(h+1)}_{\alpha,\beta}(1-u)f(xu)\,du = o(x^h) \tag{3.14}$$

as $x \to +0$ or ∞.

In Lemma 2.4 take $K(u) = \Phi^{(h)}_{\alpha-1,\beta}(1-u)$, $K(u)k(u) = \Phi^{(h+1)}_{\alpha,\beta}(1-u)$. Then it is easily verified that, for all sufficiently large values of C, $\log K(u)$ is an increasing convex function of u in (0, 1), so that $K(u)$ satisfies the condition of Lemma 2.1. Moreover, $k(u)$ and $1/k(u)$ are of bounded variation in (0, 1) for all sufficiently large C and the third condition of Lemma 2.4 is also satisfied. The result therefore follows by Lemma 2.4.

THEOREM 3.2. *To each* (α, β) *with* $\alpha > 0$ *or* $\alpha = 0$, $\beta > 0$ *corresponds a* $C_0(\alpha, \beta) \geqslant 1$ *with the following property: The existence of* (1.14) *or* (3.01) *for some* $C > C_0$ *implies its existence for every* $C > 1$.

Proof. Let C_0 be such that, for $C > C_0$, $\Phi_{\alpha-1,\beta}(1-u)$ satisfies the conditions for $K(u)$ in Lemma 2.1. Let $C_1 > C_0$ be given and let $C > 1$. Then, taking in Lemma 2.3

$$K(u) = [\Phi_{\alpha-1,\beta}(1-u)]_{C=C_1},$$
$$k(u) = \log^{\beta}\{C_1/(1-u)\}\log^{-\beta}\{C/(1-u)\}$$

and $\lambda = 0$, the theorem follows.

It follows from this theorem that in definitions 1.1 and 1.2 it is immaterial whether we say 'for an arbitrarily large C', 'for every sufficiently large C', or 'for every $C > 1$'. This means that in our definitions C is only a 'dummy' parameter.

We now proceed to prove the consistency theorems for generalized summability and continuity.

THEOREM 3.3. *If* $\sum a_n$ *is summable* (α, β) *to sum* s, *where either* $\alpha > 0$, *or* $\alpha = 0$, $\beta \geqslant 0$, *then it is summable* (α', β') *to the same sum provided either* $\alpha' > \alpha$ *or* $\alpha' = \alpha$, $\beta' > \beta$.

THEOREM 3.4. *If* $f(t) \to l$ (α, β) *as* $t \to 0$, *where either* $\alpha > 0$, *or* $\alpha = 0$, $\beta \geqslant 0$, *then* $f(t) \to l$ (α', β') *provided either* $\alpha' > \alpha$ *or* $\alpha' = \alpha$, $\beta' > \beta$.

Proof of Theorems 3.3, 3.4. We may suppose $s = 0$, $l = 0$, $t \to +0$.

(i) Suppose $\alpha = 0$, $\beta = 0$. Then, in Theorem 3.3, $A(\omega)$ is bounded for

$\omega \geqslant 0$ and $o(1)$ as $\omega \to \infty$. For $\omega \geqslant T$, where $T > 0$ is at our disposal, we have

$$\left| \int_0^1 \Phi'_{\alpha',\beta'}(1-u)A(\omega u)\,du \right| \leqslant \left| \int_0^{T/\omega} \right| + \left| \int_{T/\omega}^1 \right|$$

$$\leqslant \max_{t \geqslant 0} |A(t)| \int_0^{T/\omega} \Phi'_{\alpha',\beta'}(1-u)\,du + \max_{T \leqslant t \leqslant \omega} |A(t)| \int_0^1 \Phi'_{\alpha',\beta'}(1-u)\,du.$$

Given $\epsilon > 0$ we can choose T so large that the second term is less than $\frac{1}{2}\epsilon$; T being chosen, the first term is $o(1)$ as $\omega \to \infty$ provided $\alpha' > 0$ or $\alpha' = 0$, $\beta' > 0$. Thus

$$\int_0^1 \Phi'_{\alpha',\beta'}(1-u)A(\omega u)\,du = o(1)$$

as $\omega \to \infty$, which proves consistency for summability. The argument for continuity is precisely similar.

ii. Suppose $\alpha > 0$ or $\alpha = 0$, $\beta > 0$. Let the integer h be defined by (3.11). We have to show that for all sufficiently large C

$$\int_0^1 \Phi'_{\alpha,\beta}(1-u)f(xu)\,du = o(1) \tag{3.41}$$

implies

$$\int_0^1 \Phi'_{\alpha',\beta'}(1-u)f(xu)\,du = o(1) \tag{3.42}$$

as $x \to +0$ or ∞, provided $\alpha' > \alpha$ or $\alpha' = \alpha$, $\beta' > \beta$. If we integrate h times by parts, (3.41) becomes

$$\int_0^1 \Phi^{(h+1)}_{\alpha,\beta}(1-u)f(xu)\,du = o(1), \tag{3.43}$$

and (3.42) becomes

$$\int_0^1 \Phi^{(h+1)}_{\alpha',\beta'}(1-u)f(xu)\,du = o(1). \tag{3.44}$$

In Lemma 2.3 take $K(u) = \Phi^{(h+1)}_{\alpha,\beta}(1-u)$ and $K(u)k(u) = \Phi^{(h+1)}_{\alpha',\beta'}(u)$. Then, for sufficiently large values of C, $K(u)$ satisfies the condition of Lemma 2.1, while $k(u)$ satisfies (iii) of Lemma 2.3 in all cases. Condition (ii) of Lemma 2.3 is also satisfied, and so the lemma applies; this proves the theorem.

4. *The functions* $g_{\alpha,\beta}{}^{(t)}$, $\bar{g}_{\alpha,\beta}{}^{(t)}$.

In the theory of the summability (C, α) of Fourier series a leading

role is played by the functions

$$g_\alpha(t)=\int_0^1 (1-u)^{\alpha-1}\cos tu\,du, \qquad \bar{g}_\alpha(t)=\int_0^1 (1-u)^{\alpha-1}\sin tu\,du \quad (4.01)$$

which were first studied by W. H. Young.† In this section we introduce the generalized g-functions appropriate to the present problem.

DEFINITION 4.1. *For $C>1$, and $\alpha>0$ or $\alpha=0$, $\beta>0$,*

$$g_{\alpha,\beta}(t)=\int_0^1 \Phi_{\alpha-1,\beta}(1-u)\cos tu\,du, \qquad (4.11)$$

where $\Phi_{\alpha,\beta}(u)$ is the function (3.12), *and $\bar{g}_{\alpha,\beta}(t)$ is the function obtained on replacing* cos *by* sin *in* (4.11).

The properties of these functions that we shall employ in the sequel are all included in the following two lemmas.

LEMMA 4.2. *$g_{\alpha,\beta}(t)$, $\bar{g}_{\alpha,\beta}(t)$ and their successive derivatives are bounded in* $(0,\infty)$ *and as $t\to\infty$*

$$g_{\alpha,\beta}(t)+i\bar{g}_{\alpha,\beta}(t)=\frac{i}{t}+O\left(\frac{1}{t^2}\right)+\frac{\Gamma(\alpha)e^{i(t-\frac{\pi}{2}\alpha)}}{t^\alpha\log^\beta t}+O\left(\frac{1}{t^\alpha\log^{\beta+1}t}\right). \quad (4.21)$$

Moreover, the asymptotic formulae for the successive derivatives are obtained by differentiating (4.21) *formally.*

Considerations of space prevent us from including a proof here, but the formulae are obtained‡ from Cauchy's theorem by writing $\int_0^1=\int_0^{i\infty}-\int_1^{1+i\infty}$, and investigating these two integrals separately. The boundedness of the functions and their derivatives is an immediate consequence of their definitions.

LEMMA 4.3. *If $\alpha>1$ or $\alpha=1$, $\beta>0$, then*

$$\int_0^\infty g_{\alpha,\beta}(u)\cos tu\,du=\frac{\pi}{2}(1-t)^{\alpha-1}\log^{-\beta}\frac{C}{1-t} \qquad (0\leqslant t<1)$$

$$=0 \qquad (t\geqslant 1),$$

and

$$\int_0^\infty \bar{g}_{\alpha,\beta}(u)\sin tu\,du=\frac{\pi}{2}(1-t)^{\alpha-1}\log^{-\beta}\frac{C}{1-t} \qquad (0<t<1)$$

$$=0 \qquad (t=0,\ t\geqslant 1).$$

† Young (**17**). Young's functions were actually

$$C_\alpha(t)=\frac{t^\alpha}{\Gamma(\alpha)}\int_0^1 (1-u)^\alpha\cos tu\,du \quad (\alpha>0), \quad C_0(t)=\cos t.$$

‡ Compare Hardy (**3**), Hardy and Littlewood (**5**).

Proof. The function $(1-t)^{\alpha-1}\log^{-\beta}\{C/(1-t)\}$ is of bounded variation and continuous for $0 \leqslant t < 1$ and tends to zero as $t \to 1$. The result therefore follows from Fourier's repeated-integral formula.

5. *The two main theorems.*

We begin with two preliminary lemmas.

LEMMA 5.1. *If $\phi(t)$ is integrable-L over every finite interval $(0, T)$, $\Phi_{\alpha,\beta}(u)$ is given by* (3.12), *where $\alpha > 0$ or $\alpha = 0$, $\beta > 0$, h is defined by* (3.11) *and if, for every sufficiently large C,*

$$\int_0^1 \Phi'_{\alpha,\beta}(1-u)\phi(xu)\,du = o(1) \tag{5.11}$$

as $x \to +0$ or ∞, then

$$\int_0^\zeta \phi_h(u)\,du = o(x^{h+1}), \tag{5.12}$$

uniformly for $0 \leqslant \zeta \leqslant x$, as $x \to +0$ or ∞ respectively.†

Proof. This is simply a repetition of the proof of Theorems 3.3, 3.4 with $\alpha' = h+1$, $\beta' = 0$.

LEMMA 5.2. *If $\phi(t)$ satisfies the hypotheses of Lemma* 5.1, *then for $0 < \xi < \eta \leqslant 2\xi$,*

$$\begin{aligned}\int_\xi^\eta \phi_h(u)\,du &= o\left\{\left(\frac{\eta}{\eta-\xi}\right)^{\alpha-1-h}\log^\beta\frac{C\eta}{\eta-\xi}\,\eta^{h+1}\right\} \text{ in case (i) of (3.11)} \\ &= o\left\{\left(\frac{\eta}{\eta-\xi}\right)^{-1}\log^{\beta+1}\frac{C\eta}{\eta-\xi}\,\eta^{h+1}\right\} \text{ in case (ii) of (3.11)}\end{aligned} \tag{5.21}$$

uniformly in η as $\xi \to +0$ or ∞ respectively.

Proof. We give the argument in case (i); that in case (ii) is substantially the same. Since, for every sufficiently large C, $u^{h+1-\alpha}\log^\beta(C/u)$ increases in $(0, 1)$, we have by the second mean-value theorem

$$\begin{aligned}\int_\xi^\eta \phi_h(u)\,du &= \int_0^{\eta-\xi} \phi_h(\eta-u)\,du = \eta\int_0^{(\eta-\xi)/\eta} \phi_h\{\eta(1-u)\}\,du \\ &= \eta\int_0^{(\eta-\xi)/\eta} u^{h+1-\alpha}\log^\beta\frac{C}{u}\,.\,u^{\alpha-1-h}\log^{-\beta}\frac{C}{u}\,\phi_h\{\eta(1-u)\}\,du \\ &= \eta\left(\frac{\eta-\xi}{\eta}\right)^{h+1-\alpha}\log^\beta\frac{C\eta}{\eta-\xi}\int_\zeta^{(\eta-\xi)/\eta} u^{\alpha-1-h}\log^{-\beta}\frac{C}{u}\,\phi_h\{\eta(1-u)\}\,du,\end{aligned}$$

† Where, as usual, $\phi_h(t) = \int_0^t \phi_{h-1}(u)\,du$, $\phi_0(t) = \phi(t)$.

where $0 \leqslant \zeta < (\eta-\xi)/\eta$,

$$= \eta\left(\frac{\eta}{\eta-\xi}\right)^{\alpha-1-h} \log^{\beta}\frac{C\eta}{\eta-\xi} \int_{\xi/\eta}^{1-\zeta} (1-u)^{\alpha-1-h} \log^{-\beta}\frac{C}{1-u}\phi_h(\eta u)\,du. \quad (5.22)$$

Integrating (5.11) by parts h times, we have

$$\int_0^1 \Phi_{\alpha,\beta}^{(h+1)}(1-u)\phi_h(xu)\,du = o(x^h) \quad (5.23)$$

as $x \to +0$ or ∞. Since, for sufficiently large C, $\Phi_{\alpha,\beta}^{(h+1)}(1-u)$ satisfies the condition for $K(u)$ in Lemma 2.1, we infer from Lemma 2.3 that

$$\int_0^1 (1-u)^{\alpha-1-h} \log^{-\beta}\frac{C}{1-u}\phi_h(\eta u)\,du = o(\eta^h) \quad (5.24)$$

as $\eta \to +0$ or ∞. The kernel $(1-u)^{\alpha-1-h}\log^{-\beta}\dfrac{C}{1-u}$ also satisfies the hypotheses of Lemma 2.1, as we easily verify. It follows from Lemma 2.1 that

$$\int_{\zeta_1}^{\zeta_2} (1-u)^{\alpha-1-h} \log^{-\beta}\frac{C}{1-u}\phi_h(\eta u)\,du = o(\eta^h)$$

uniformly for $0 \leqslant \zeta_1 \leqslant \zeta_2 \leqslant 1$ as $\eta \to +0$ or ∞, or, which is the same thing, as $\xi \to +0$ or ∞. Taking $\zeta_1 = \xi/\eta$, $\zeta_2 = 1-\zeta$, we see that (5.22) is

$$o\left\{\left(\frac{\eta}{\eta-\xi}\right)^{\alpha-1-h} \log^{\beta}\frac{C\eta}{\eta-\xi}\eta^{h+1}\right\}$$

uniformly in η, as $\xi \to +0$ or ∞.

THEOREM 5.3. *If $f(t)$ is integrable-L in $(-\pi, \pi)$ with period 2π, and if*

$$\phi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)-2s\} \to 0 \quad (\alpha, \beta) \quad (5.31)$$

as $t \to 0$, where either $\alpha > 0$, or $\alpha = 0$, $\beta \geqslant 0$, then the Fourier series of $f(t)$ is summable (α, γ) to sum s at $t = x$ for every $\gamma > \beta+1+\varpi$, where

$$\begin{aligned} \varpi &= 0 \quad \text{if } \alpha = 0, \beta = 0 \text{ and in case (i) of (3.11)} \\ &= 1 \quad \text{in case (ii) of (3.11).} \end{aligned} \quad (5.32)$$

Proof. Let

$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{m=1}^{\infty}(a_m \cos mx + b_m \sin mx)$$

$$= \tfrac{1}{2}a_0 + \sum_{m=0}^{\infty} c_m(x), \quad (5.33)$$

so that

$$\phi(t) \sim (\tfrac{1}{2}a_0 - s) + \sum_{m=1}^{\infty} c_m(x)\cos mt.$$

It follows from Lemma 4.2 that for either $\alpha > 0$, or $\alpha = 0$, $\gamma > 1$, $g'_{1+\alpha,\gamma}(t)$ is absolutely integrable in $(0, \infty)$ and tends to zero as $t \to \infty$. Hence†

$$\frac{2}{\pi}\int_0^\infty g_{1+\alpha,\gamma}(t)\,\phi(t/\omega)\,dt$$

$$= \frac{2}{\pi}(\tfrac{1}{2}a_0 - s)\int_0^\infty g_{1+\alpha,\gamma}(t)\,dt + \frac{2}{\pi}\sum_{m=1}^{\infty} c_m(x)\int_0^\infty g_{1+\alpha,\gamma}(t)\cos\frac{mt}{\omega}\,dt$$

$$= (\tfrac{1}{2}a_0 - s) + \sum_{1\leqslant m<\omega} B\left(1-\frac{m}{\omega}\right)^{\alpha}\log^{-\beta}\frac{C}{1-(m/\omega)}\,c_m(x).$$

Thus for $\alpha > 0$ or $\alpha = 0$, $\gamma > 1$ the summability (α, γ) of the Fourier series (5.33) to sum s at the point x is equivalent to

$$\int_0^\infty g_{1+\alpha,\gamma}(t)\,\phi(t/\omega)\,dt = o(1) \tag{5.34}$$

as $\omega \to \infty$, i.e.

$$\omega\int_0^\infty g_{\alpha+1,\gamma}(\omega u)\,\phi(u)\,du = o(1) \tag{5.35}$$

as $\omega \to \infty$.

The case $\alpha = 0$, $\beta = 0$ has been considered elsewhere.‡ We therefore confine ourselves to the case $\alpha > 0$ or $\alpha = 0, \beta > 0$. We have to show that (5.34) is a consequence of

$$\int_0^1 \Phi'_{\alpha,\beta}(1-u)\,\phi(tu)\,du = o(1)$$

as $t \to 0$, provided $\gamma > \beta + 1 + \varpi$. Given $\epsilon > 0$ we can, by Lemma 5.2 with $x \to +0$, choose $\delta > 0$, so that for $0 < \xi < \eta \leqslant 2\xi < \delta$

$$\left|\int_\xi^\eta \phi_h(u)\,du\right| < \epsilon\eta^{h+1}\left(\frac{\eta}{\eta-\xi}\right)^{\alpha-1-h}\log^{\beta+\varpi}\frac{C\eta}{\eta-\xi}. \tag{5.36}$$

Write

$$\omega\int_0^\infty g_{\alpha+1,\gamma}(\omega u)\,\phi(u)\,du = \omega\int_0^\delta + \omega\int_\delta^\infty = J^* + J^{**}.$$

Then, for every $\delta > 0$, $J^{**} \to 0$ as $\omega \to \infty$. For by Lemma 4.2

$$g_{\alpha+1,\gamma}(t) = O(t^{-2}) + O(t^{-\alpha-1}\log^{-\gamma}t) = O(t^{-1}\log^{-1-k}t)$$

† By a well-known theorem. See Hobson (**2**), p. 583.

‡ Bosanquet and Linfoot (**2**).

for large t, where $k>0$ provided $\alpha>0$ or $\alpha=0$, $\gamma>1$. Thus, writing

$$\Phi^*(t)=\int_0^t |\phi(u)|\,du,$$ we have for $\omega>1/\delta$

$$|J^{**}|<A\int_\delta^\infty u^{-1}\log^{-1-k}(\omega u)|\phi(u)|\,du$$

$$<A\left|[u^{-1}\log^{-1-k}\omega u\,\Phi^*(u)]_\delta^\infty\right|+A\left|\int_\delta^\infty u^{-2}\log^{-1-k}(\omega u)\,\Phi^*(u)\,du\right|,$$

where the A's are independent of ω,

$$=A\delta^{-1}\log^{-1-k}(\omega\delta)\,\Phi^*(\delta)+\int_\delta^\infty u^{-2}\log^{-1-k}\omega u\,O(u)\,du$$

$$=O(\log^{-1-k}\omega)+O(\log^{-k}\omega)=o(1)$$

as $\omega\to\infty$. Again, integrating J^* by parts h times, we have

$$J^*=\sum_{\nu=0}^{h-1}(-1)^\nu\omega^{\nu+1}[g^{(\nu)}_{\alpha+1,\gamma}(\omega u)\phi_{\nu+1}(u)]_0^\delta+(-1)^h\omega^{h+1}\int_0^\delta g^{(h)}_{\alpha+1,\gamma}(\omega u)\phi_h(u)\,du,$$

and the integrated terms are all $o(1)$ as $\omega\to\infty$, by Lemma 4.2. Now let $\tau=[\omega\delta]$ and write

$$\omega^{h+1}\int_0^\delta g_{\alpha+1,\gamma}(\omega u)\phi_h(u)\,du=\omega^{h+1}\int_0^{1/\omega}+\sum_{\nu=1}^{\tau-1}\omega^{h+1}\int_{\nu/\omega}^{(\nu+1)/\omega}+\omega^{h+1}\int_{\tau/\omega}^\delta$$

$$=J_0+\sum_{\nu=1}^{\tau-1}J_\nu+J_\tau. \tag{5.37}$$

By Lemma 4.2, $g^{(h)}_{\alpha+1,\gamma}(t)$ and $g^{(h+1)}_{\alpha+1,\gamma}(t)$ are both bounded in $(0,\infty)$ and $O(t^{-\alpha-1}\log^{-\gamma}t)$ as $t\to\infty$, so that $g^{(h)}_{\alpha+1,\gamma}(t)$ is of bounded variation in $(0,\infty)$, and in any interval $(\nu,\nu+1)$ is the difference of two positive decreasing functions each of which is $O(\nu^{-\alpha-1}\log^{-\gamma}C\nu)$ as $\nu\to\infty$. Thus

$$|J_0|<A\omega^{h+1}\max_{0\leqslant\lambda<\mu\leqslant1/\omega}\left|\int_\lambda^\mu\phi_h(u)\,du\right|=\omega^{h+1}o\left(\frac{1}{\omega^{h+1}}\right)=o(1),$$

and, for $\nu\geqslant1$,

$$|J_\nu|<A\omega^{h+1}\nu^{-\alpha-1}\log^{-\gamma}C\nu\max_{\nu\leqslant\lambda<\mu\leqslant\nu+1}\left|\int_{\lambda/\omega}^{\mu/\omega}\phi_h(u)\,du\right|,$$

where the A are independent of ω, ν. Taking $\xi=\nu/\omega$, $\eta=(\nu+1)/\omega$ in (5.21), so that $\eta/(\eta-\xi)=\nu$, we have for every sufficiently large C

$$|J_\nu| < A\omega^{h+1}\nu^{-\alpha-1}\log^{-\gamma}C\nu \cdot \epsilon\left(\frac{\nu+1}{\omega}\right)^{h+1}\nu^{\alpha-1-h}\log^{\beta+\varpi}C\nu$$
$$= A\epsilon\nu^{-1}\log^{\beta+\varpi-\gamma}C\nu.$$

A precisely similar argument shows that, for every sufficiently large C, $|J_\tau| < A\epsilon\tau^{-1}\log^{\beta+\varpi-\gamma}(C\tau)$. Thus

$$|J^*| < o(1)+o(1)+A\epsilon\sum_{\nu=1}^{\tau}\nu^{-1}\log^{\beta+\varpi-\gamma}(C\nu)$$
$$< o(1)+A\epsilon\sum_{\nu=1}^{\infty}\nu^{-1}\log^{\beta+\varpi-\gamma}(C\nu)$$

for $\gamma > \beta+1+\varpi$, and so $J^* \to 0$ as $\omega \to \infty$. This completes the proof.

THEOREM 5.4. *If $f(t)$ is integrable-L in $(-\pi, \pi)$ with period 2π, and if at the point $t = x$ the Fourier series of $f(t)$ is summable (α, β) to sum s, where $\alpha > 0$ or $\alpha = 0$, $\beta \geqslant 0$, then*

$$\phi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)-2s\} \to 0 \quad (\alpha+1, \gamma) \tag{5.41}$$

as $t \to 0$ for every $\gamma > \beta+1+\varpi$, where ϖ is defined as in Theorem 5.3.

Proof. We may suppose $s = \frac{1}{2}a_0 = 0$ without loss of generality. By (1.2) our hypotheses can be written

$$\int_0^1 \Phi'_{\alpha,\beta}(1-u)A(\omega u)\,du \to 0 \qquad (\alpha > 0;\ \alpha = 0, \beta > 0)$$
$$A(\omega) \to 0 \qquad (\alpha = 0, \beta = 0) \tag{5.42}$$

as $\omega \to \infty$, and our conclusion, which, since ϕ is even, can be stated as

$$\int_0^1 \Phi_{\alpha,\gamma}(1-u)\phi(tu)\,du \to 0 \tag{5.43}$$

as $t \to +0$, may, since $\Phi_{\alpha,\gamma}(1-u)$ is of bounded variation in $(0, 1)$, be written†

$$\sum_{n=1}^{\infty} C_n(t)\int_0^1 \Phi_{\alpha,\gamma}(1-u)\cos ntu\,du \to 0 \tag{5.44}$$

as $t \to +0$, i.e.

$$\sum_{n=1}^{\infty} C_n g_{\alpha+1,\gamma}(nt) \to 0 \tag{5.45}$$

as $t \to +0$. Since $A(n)g_{\alpha+1,\gamma}(nt) = o(n)O(n^{-1}) = o(1)$ as $n \to \infty$, the sum in (5.44) may, by Lemma 4.2, be written

$$\sum_{n=1}^{\infty}\{A(n)-A(n-1)\}g_{\alpha+1,\gamma}(nt) = \sum_{n=1}^{\infty}A(n)\{g_{\alpha+1,\gamma}(nt)-g_{\alpha+1,\gamma}[(n+1)t]\}$$
$$= -\sum_{n=1}^{\infty}A(n)\int_n^{n+1} g'_{\alpha+1,\gamma}(tu)\,du = -\int_0^{\infty} g'_{\alpha+1,\gamma}(tu)A(u)\,du.$$

† By the theorem quoted in the proof of Theorem 5.3.

Thus we have to prove that (5.42) implies

$$\int_0^\infty g'_{\alpha+1,\gamma}(u)A(\omega u)\,du \to 0 \tag{5.46}$$

as $\omega \to \infty$.

Consider first the case $\alpha = 0$, $\beta = 0$. We shall prove that (5.42) implies (5.43) for $\gamma > 1$. Given $\epsilon > 0$, we can, since $A(t)$, $g'_{1,\gamma}(t)$ are bounded in $(0, \infty)$, choose $\delta > 0$, so that

$$\int_0^\delta g'_{1,\gamma}(u)A(\omega u)\,du < \tfrac{1}{2}\epsilon$$

for all $\omega > 0$. Then δ, ϵ being fixed,

$$\left|\int_\delta^\infty g'_{1,\gamma}(u)A(\omega u)\,du\right| \leqslant \max_{t\leqslant\omega\delta}|A(t)|\int_\delta^\infty |g'_{1,\gamma}(u)|\,du < \tfrac{1}{2}\epsilon$$

for $\omega > \omega_0(\epsilon)$, since the integral converges for $\gamma > 1$. Thus if $\gamma > 1$,

$$\left|\int_0^\infty g'_{1,\gamma}(u)A(\omega u)\,du\right| < \epsilon \tag{5.47}$$

for $\epsilon > 0$, $\omega > \omega_0(\epsilon)$.

Next suppose $\alpha > 0$ or $\alpha = 0$, $\beta > 0$. Integrating the left-hand side of (5.46) by parts h times, we obtain

$$\int_0^\infty g'_{\alpha+1,\gamma}(u)A(\omega u)\,du$$

$$= \sum_{\nu=1}^{h}\frac{(-1)^{\nu-1}}{\omega^\nu}\left[g^{(\nu)}_{\alpha+1,\gamma}(u)A_\nu(\omega u)\right]_0^\infty + \frac{(-1)^h}{\omega^h}\int_0^\infty g^{(h+1)}_{\alpha+1,\gamma}(u)A_h(\omega u)\,du.$$

Since $A_0(t) = A(t) = o(t)$ as $t \to \infty$, $A_\nu(\omega u) = o(u^{\nu+1})$ as $u \to \infty$ for every $\omega > 0$. Moreover by Lemma 4.2

$$g^{(\nu)}_{\alpha+1,\gamma}(u) = O(u^{-\nu-2}) + O(u^{-\alpha-1}\log^{-\gamma}Cu)$$

as $u \to \infty$ and so the integrated terms vanish for $\gamma > \beta + 1 + \varpi$. Thus

$$\int_0^\infty g'_{\alpha+1,\gamma}(u)A(\omega u)\,du = \frac{(-1)^h}{\omega^h}\int_0^\infty g^{(h+1)}_{\alpha+1,\gamma}(u)A_h(\omega u)\,du. \tag{5.48}$$

Given $\epsilon > 0$ we can, by Lemma 5.2, choose $T > 0$ so that for $T < \xi < \eta \leqslant 2\xi$

$$\left|\int_\xi^\eta A_h(u)\,du\right| < \epsilon\eta^{h+1}\left(\frac{\eta}{\eta-\xi}\right)^{\alpha-1-h}\log^{\beta+\varpi}\frac{C\eta}{\eta-\xi} \tag{5.49}$$

Then writing

$$\frac{(-1)^h}{\omega^h}\int_0^\infty g_{\alpha+1,\gamma}^{(h+1)}(u)A_h(\omega u)\,du=\frac{(-1)^h}{\omega^h}\int_0^T+\frac{(-1)^h}{\omega^h}\int_T^\infty=J_1+J_2,$$

we see in the first place that $J_1=o(1)$ as $\omega\to\infty$. For since $g_{\alpha+1,\gamma}^{(h+1)}(u)$ is the difference of two bounded positive decreasing functions in $(0,\infty)$,

$$|J_1|<\frac{A}{\omega^h}\max_{0\leqslant\lambda<\mu\leqslant T}\left|\int_\lambda^\mu A_h(\omega u)\,du\right|=O\left(\frac{1}{\omega^h}\right)\frac{1}{\omega}\,o(\omega^{h+1})=o(1),$$

by Lemma 5.1. Again

$$J_2=\sum_{\nu=1}^\infty\frac{(-1)^h}{\omega^h}\int_{T+\nu-1}^{T+\nu}g_{\alpha+1,\gamma}^{(h+1)}(u)A_h(\omega u)\,du.$$

By Lemma 4.2, $g_{\alpha+1,\gamma}^{(h+1)}(u)$ is in $(T+\nu-1,\ T+\nu)$ the difference of two positive decreasing functions, each of which is $O(\nu^{-\alpha-1}\log^{-\gamma}C\nu)$ as $\nu\to\infty$. Hence, as in the last theorem,

$$\begin{aligned}&\left|\frac{(-1)^h}{\omega^h}\int_{T+\nu-1}^{T+\nu}g_{\alpha+1,\gamma}^{(h+1)}(u)A_h(\omega u)\,du\right|\\&\quad<\frac{A}{\omega^h}\nu^{-\alpha-1}\log^{-\gamma}(C\nu)\max_{\nu-1\leqslant\lambda<\mu\leqslant\nu}\left|\int_{T+\lambda}^{T+\mu}A_h(\omega u)\,du\right|\\&\quad<\frac{A}{\omega^h}\nu^{-\alpha-1}\log^{-\nu}(C\nu)\,.\,\frac{1}{\omega}\epsilon(T+\nu)^{h+1}(T+\nu)^{\alpha-1-h}\log^{\beta+\varpi}\{C(T+\nu)\}\\&\quad<A\epsilon\nu^{-1}\log^{\beta+\varpi+\gamma}(C\nu)\end{aligned}$$

for $\nu\geqslant 1$, $\omega\geqslant 1$, and all sufficiently large C. Thus

$$|J_2|<A\epsilon\sum_{\nu=1}^\infty\nu^{-1}\log^{\beta+\varpi-\gamma}(C\nu)$$

for $\epsilon>0$, $\omega\geqslant 1$, $T>T_0(\epsilon)$ and all sufficiently large C. Since the series converges for $\gamma>\beta+1+\varpi$, it follows that for such a γ

$$\int_0^\infty g'_{\alpha+1,\gamma}(u)A(\omega u)\,du\to 0$$

as $\omega\to\infty$ and the theorem is proved.

6. In this section we discuss briefly the corresponding problem for allied series, and give the theorems analogous to Theorems C and D of the introduction.

THEOREM 6.1. *If $f(t)$ is integrable-L in $(-\pi, \pi)$ with period 2π, and if*

$$\psi(t) = \tfrac{1}{2}\{f(x+t)-f(x-t)\} \to 0 \quad (\alpha, \beta) \tag{6.11}$$

as $t \to 0$, where either $\alpha > 0$, or $\alpha = 0$, $\beta > 0$, then the allied series of $f(t)$ is summable (α, γ) at $t = x$ for every $\gamma > \beta+1+\varpi$ where ϖ is defined as in Theorem 5.3, to the sum

$$\lim_{t\to 0} \frac{2}{\pi} \int_t^\infty \frac{\psi(u)}{u}\, du \tag{6.12}$$

provided this exists† by some Cesàro mean.

Proof. Writing

$$\psi(t) = \tfrac{1}{2}\{f(x+t)-f(x-t)\} \sim \sum_{m=1}^{\infty} (b_m \cos mx - a_m \sin mx) \sin mt$$

$$= \sum_{m=1}^{\infty} \bar{c}_m(x) \sin mt,$$

we deduce from Lemma 4.3, as in the proof of Theorem 5.3, that the summability (α, γ) of the allied series $\sum \bar{c}_m(x)$ to sum s is equivalent to

$$\frac{2\omega}{\pi} \int_0^\infty \bar{g}_{\alpha+1,\gamma}(\omega u)\psi(u)\, du \to s \tag{6.13}$$

as $\omega \to \infty$. The rest of the proof is similar to that of Theorem 5.3, $g^{(h)}_{\alpha+1,\gamma}(\omega u)$ being replaced by the function

$$\bar{g}^{(h)}_{\alpha+1,\gamma}(\omega u) - \frac{(-1)^h h!}{\omega^{h+1} u^{h+1}}$$

in the interval $(1/\omega, \infty)$. There is an additional term

$$\frac{2}{\pi} h! \int_{1/\omega}^\infty \frac{\psi_h(u)}{u^{h+1}}\, du; \tag{6.14}$$

by induction it tends to s as $\omega \to \infty$, which gives the desired result.

THEOREM 6.2. *If $f(t)$ is integrable-L in $(-\pi, \pi)$ with period 2π, and if the allied series of $f(t)$ is summable (α, β) to s at $t = x$, where either $\alpha > 0$, or $\alpha = 0$, $\beta > 0$, then*

$$\psi(t) = \tfrac{1}{2}\{f(x+t)-f(x-t)\} \to 0 \quad (\alpha+1, \gamma) \tag{6.21}$$

as $t \to 0$ for every $\gamma > \beta+1+\varpi$, where ϖ is defined as in Theorem 5.3, and the limit

$$\lim_{t\to 0} \frac{2}{\pi} \int_t^\infty \frac{\psi(u)}{u}\, du \tag{6.22}$$

exists by some Cesàro mean and has the value s.

† The integrals in (6.12), (6.22) are in general Cauchy-Lebesgue integrals.

Proof. Supposing for simplicity that $s=0$, we may write our hypothesis in the form

$$\int_0^1 \Phi_{\alpha,\beta}(1-u)\bar{A}(\omega u)\,du \to 0 \qquad (\alpha>0;\ \alpha=0,\beta>0) \tag{6.23}$$

$$\bar{A}(\omega)\to 0 \qquad (\alpha=0,\beta=0)$$

as $\omega\to\infty$, where

$$\bar{A}(\omega)=\sum_{m\leqslant\omega}\bar{c}_m(x). \tag{6.24}$$

We have to show that for $\gamma>\beta+1+\varpi$ this implies

$$\int_0^1 \Phi_{\alpha,\gamma}(1-u)\psi(tu)\,du\to 0 \tag{6.25}$$

and

$$\frac{2}{\pi}\int_t^\infty \frac{\psi(u)}{u}\,du\to 0 \quad (C) \tag{6.26}$$

as $t\to+0$, since ψ is an odd function. As in the proof of Theorem 5.4, we can write (6.25) in the form

$$\int_0^\infty \bar{g}'_{\alpha+1,\gamma}(u)\bar{A}(\omega u)\,du\to 0 \tag{6.27}$$

as $\omega\to\infty$. The argument then follows that of Theorem 5.4.

Finally (6.26) follows at once from Theorem D of the introduction, since our hypotheses ensure that the allied series is summable $(C, h+1)$.

REFERENCES

1. L. S. Bosanquet, 'On the summability of Fourier series': *Proc. London Math. Soc.* (2) 31 (1930), 144–64.
2. L. S. Bosanquet and E. H. Linfoot, 'On the zero order summability of Fourier series': *Jour. London Math. Soc.* 6 (1931), 117–26.
3. G. H. Hardy, 'On the zeros of certain classes of integral Taylor series, II': *Proc. London Math. Soc.* (2) 2 (1904), 401–31.
4. G. H. Hardy and J. E. Littlewood, 'Solution of the Cesàro summability problem for power series and Fourier series': *Math. Zeitschrift*, 19 (1924), 67–96.
5. —— 'The Allied series of a Fourier series': *Proc. London Math. Soc.* (2) 24 (1926), 211–46.
6. —— 'Notes on the theory of series (III): On the summability of the Fourier series of a nearly continuous function': *Proc. Cambridge Phil. Soc.*, 23 (1927), 681–4.
7. —— 'Notes on the theory of series (VII): On Young's convergence criterion for Fourier series': *Proc. London Math. Soc.* (2) 28 (1928), 301–11.

8. G. H. Hardy and M. Riesz, 'The general theory of Dirichlet's series': *Cambridge Math. Tract*, 18 (1915).
9. E. W. Hobson, *The Theory of Functions of a Real Variable*, 2nd ed. (1926).
10. P. Lévy, *Bull. des Sciences Math.*, 47 (1923), 343–52.
11. R. E. A. C. Paley, 'On the Cesàro summability of Fourier series and allied series': *Proc. Cambridge Phil. Soc.*, 26 (1930), 173–203.
12. S. Pollard, 'The summation of Denjoy-Fourier series': *Proc. London Math. Soc.* (2) 27 (1928), 209–22.
13. M. Riesz, 'Une méthode de sommation' etc.: *Comptes rendus*, 152 (1911), 1651–4.
14. —— 'Sur un théorème de la moyenne et ses applications': *Acta Univ. Hungaricae Szeged.* (1923), 114–26.
15. S. Verblunsky, 'Note on the sum of an oscillating series': *Proc. Cambridge Phil. Soc.*, 26 (1930), 152–7.
16. ——'On the limit of a function at a point'. *Proc. London Math. Soc.* (2) 32 (1931), 163–99.
17. W. H. Young, 'On infinite integrals involving a generalization of the sine and cosine functions': *Quart. J. of Math.*, 43 (1912), 161–77.
18. A. Zygmund, *Comptes rendus*, 179 (1924), 870–2.

A CLASS OF ARITHMETICAL IDENTITIES*

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1. Foreword. The identities discussed in this note require no previous knowledge for their comprehension. We employ only elementary algebra. To illustrate the type of identity I quote the following result due to Uspensky.†

Suppose that $F(X, Y, Z)$ is any function of X, Y, Z such that

$$F(-X, Y, Z) = -F(X, Y, Z), \quad F(X, -Y, -Z) = F(X, Y, Z). \quad (1.1)$$

Then

$$\sum F(y+z, x, y-z) = 2 \sum F(z-2x, x+y, 2x+2y-z) \quad (1.2)$$

where in each sum x, y, z range over integral values such that

$$x^2+yz = n, \qquad y > 0, \qquad z > 0, \quad (1.3)$$

and n is a given number which is not a square. If n is a square, a certain easy modification is necessary.

In a recent note‡ Uspensky has given some new formulae which were obtained, he remarks, by 'groping in the dark'. Once found their proof is simple.

It may not be out of place therefore (i) to give a general identity which will include identities such as these, and (ii) to show that the condition (1.1) on F is needlessly restrictive.

These identities, we may note, have been applied with much success, notably by Nazimov and Uspensky, to the study of class-relation formulae and the number of representations of a number by certain quadratic forms.

In this note I confine myself, however, to the establishment of the identity which follows.

* I am indebted to the referees for various suggestions.

† For references see L. J. Mordell, 'On an arithmetical formula', *Jour. London Math. Soc.*, 4 (1929), 291–6.

‡ J. V. Uspensky, 'On incomplete numerical functions': *Bull. American Math. Soc.*, 36 (1930), 743–54. There are three misprints on p. 748. Read $\Delta'+3\Delta-2h$, $\Delta-\Delta'-2h$, $\Delta'-9\Delta-6h$ for $\Delta'+3\Delta-h$, $\Delta'-\Delta-2h$, $\Delta'-\Delta-6h$ respectively.

2. Statement of the identity. *Let μ, ν be any fixed real numbers, not necessarily integers. Let $F(X,Y,Z)$, $\phi(X,Y,Z)$ be any functions of X,Y,Z subject to the conditions*

$$F(X,Y,Z)+F(X,-Y,-Z)+ \\ +F(-X,Y,Z)+F(-X,-Y,-Z)=0, \quad (2.11)$$

$$\phi(X,-Y,-Z)=\phi(X,Y,Z), \quad \phi(-X,Y,Z)=\phi(X,Y,Z).^* \quad (2.12)$$

Let

$$S=\sum F(y+z,z-y,x) \quad (2.2)$$

over integers x, y, z such that

$$\phi(y+z,z-y,x)=n, \quad y>0, \quad z>0, \quad \nu x+\mu y-\mu z\neq 0, \quad (2.21)$$

where n is any given number.

Let

$$S'=\sum\{F(\xi,\eta,\zeta)+F(\xi,-\eta,-\zeta)\} \quad (2.3)$$

over integers ξ, η, ζ such that

$$\phi(\xi,\eta,\zeta)=n, \quad \xi\equiv\eta \pmod 2, \quad \xi>\eta, \quad \mu\eta>\nu\zeta, \quad (2.31)$$

$$\xi+\eta\neq 0. \quad (2.32)$$

Under these conditions we shall have

$$S=S'. \quad (2.4)$$

Before proceeding to the proof, we observe that in Uspensky's work we have essentially

$$\phi(X,Y,Z)=X^2-Y^2+Z^2, \qquad \phi(y+z,z-y,x)=x^2+4yz.$$

To obtain contact between our identity and those of Uspensky it is necessary to employ a special case of the transformation

$$\mu(\xi+\eta)=2(x'-\lambda y'+z'), \qquad \xi-\eta=2y', \qquad \nu\zeta=x'-2\lambda y',$$

which is such that

$$\lambda\mu(\xi^2-\eta^2)+\nu^2\zeta^2=x'^2+4\lambda y'z'.$$

Herein μ and ν are now integers and λ is an integer; x', y', z' are integers such that ξ, η, ζ satisfy the conditions above. For Uspensky's new identities take $\lambda=1$, $\mu=\nu=4$ and $\lambda=1$, $\mu=\nu=2$.

One more observation may not be unnecessary. The solutions of $\phi=n$ may be infinite. In that case we naturally assume that F is such that each of the three infinite series arising in S and S' is absolutely convergent. More simply we may suppose that $F(X,Y,Z)$ is zero outside the cube whose vertices are $(\pm R,\pm R,\pm R)$ where R is any assigned number.

* The most general ϕ of the second degree is $aX^2+bY^2+2fYZ+cZ^2+k$, where a, b,..., k are any constants.

3. Proof of (2.4). The functions F and ϕ being well-nigh arbitrary, it is evident that in order to prove our identity we must set up certain one-one correspondences between the solutions of $\phi(y+z, z-y, x) = n$ and the solutions of $\phi(\xi, \eta, \zeta) = n$; and these relations plainly must be of the form

$$\xi = y+z, \qquad \eta = z-y, \qquad \zeta = x \tag{3.11}$$

or

$$\xi' = y+z, \qquad \eta' = y-z, \qquad \zeta' = -x \tag{3.12}$$

since $(-x, z, y)$ is a solution of $\phi(y+z, z-y, x) = n$ when (x, y, z) is one.

We apply (3.11) to those solutions (x, y, z) which satisfy

$$y > 0, z > 0, \qquad \nu x + \mu y < \mu z. \tag{3.2}$$

The corresponding ξ, η, ζ are integers such that

$$\phi(\xi, \eta, \zeta) = n, \qquad \xi \equiv \eta \pmod 2, \qquad \xi > \eta, \qquad \mu\eta > \nu\zeta, \tag{3.21}$$

$$\xi + \eta > 0. \tag{3.22}$$

Conversely, if (ξ, η, ζ) satisfy (3.21) and (3.22), then (x, y, z) satisfy (3.2).

We apply (3.12) to those solutions (x, y, z) which satisfy

$$y > 0, z > 0, \qquad \nu x + \mu y > \mu z. \tag{3.3}$$

Then the corresponding ξ', η', ζ' satisfy the *same* equations as do ξ, η, ζ in (3.21) and (3.22). Since (3.2) and (3.3) are equivalent to (2.21), it follows immediately that

$$S' = \sum \{F(\xi, \eta, \zeta) + F(\xi, -\eta, -\zeta)\} \tag{3.4}$$

extended over integers ξ, η, ζ subject to (3.21) and (3.22).

The identity (2.4) will result provided we can replace the condition $\xi + \eta > 0$ by $\xi + \eta \neq 0$. Let $\overline{S}$ denote the sum (3.4) extended over ξ, η, ζ subject to (3.21) and

$$\xi + \eta < 0. \tag{3.22'}$$

From a set ξ, η, ζ satisfying (3.21) and (3.22′) we obtain another set ξ', η', ζ' satisfying the *same* conditions by means of the self-reciprocal substitution determined by

$$\xi' = -\xi, \qquad \eta' = \eta, \qquad \zeta' = \zeta \tag{3.5}$$

which yields

$$\xi' + \eta' = -\xi + \eta < 0, \qquad -\xi' + \eta' = \xi + \eta < 0. \tag{3.51}$$

Since $\phi(-X, Y, Z) = \phi(X, Y, Z)$, we see that as (ξ, η, ζ) runs over the set of integers which satisfy (3.21) and (3.22′), so also does

(ξ', η', ζ'). And consequently, the sums being subject to (3.21) and (3.22′),

$$\begin{aligned}\bar{S} &= \sum \{F(\xi, \eta, \zeta) + F(\xi, -\eta, -\zeta)\} \\ &= \sum \{F(-\xi', \eta', \zeta') + F(-\xi', -\eta', -\zeta')\} = -S\end{aligned}$$

in virtue of the condition (2.11) on F, so that

$$\bar{S} = 0. \tag{3.6}$$

The identity (2.4) follows from (3.4) and (3.6).

4. Summary. It will have been observed that the linear forms $y+z$, $z-y$, x occur constantly. The reason is not far to seek. The matrix of the substitution $X = -x$, $Y = z$, $Z = y$ has latent roots $1, -1, -1$. It follows that just one linear form is unchanged by the substitution and that an infinity of linear forms are changed simply in sign: the former is $y+z$, and the latter are linear combinations of x and $z-y$.

It is these facts that have been essential from the algebraic point of view in the preceding analysis. The rest of the analysis is dictated by elementary considerations of number theory.

PARTITION-GENERATING FUNCTIONS

By T. W. CHAUNDY

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MR. E. M. WRIGHT has called attention in this number of the *Quarterly Journal* to MacMahon's theory of 'plane partitions', and in particular to the absence of any 'intuitive' proof of MacMahon's generating function of plane partitions*

$$\prod_{r=1}^{\infty}(1-x^r)^{-r}.$$

From the nature of the problem an intuitive proof seems unlikely, but it is possible to devise an algebraic technique which gives the result more rapidly than MacMahon's own method of "lattice functions".

MacMahon made a fundamental contribution to partition-theory in prescribing the monotonic order of parts in the partition, and the gist of the method here to be described lies precisely in the embodiment, in algebraic form, of this monotonic condition. The principle of the method is most simply seen, if we apply it in the first place to the linear partition $aa'a''\ldots$, arranged, of course, in non-ascending order.

Consider algebraic functions ξ_a defined for all positive integers (and zero) a by the recurrence-formula

$$\xi_a = \sum_{r=0}^{a} x^r \xi_r. \tag{1}$$

Out of the infinite series

$$\xi_\infty = \sum_{a=0}^{\infty} x^a \xi_a, \tag{2}$$

regarded as a generating function, we can build up a power x^n in the following way. In (2) pick out any power x^a: this brings with it its coefficient ξ_a in which, by virtue of (1), we can pick out any power $x^{a'}$, where $a' \leqslant a$. This again brings its coefficient $\xi_{a'}$, in which we can pick out any power $x^{a''}$, where $a'' \leqslant a'$. We can proceed in this way, not actually *ad infinitum*, since a finite x^n is prescribed, but certainly *ad libitum*.

It is evident that, when at length we have built up the power x^n, we have obtained *pari passu* the partition $aa'a''\ldots$ of n arranged *in*

* *Combinatory Analysis* (1916), vol. ii, p. 243, § 495, the last formula.

non-ascending order. Thus ξ_∞ is the generating function of such partitions and, in particular, ξ_a is the generating function of such partitions in which the first (and therefore each) part may not exceed a.

To clinch the argument let us solve the equations (1). Subtraction gives

$$\xi_a - \xi_{a-1} = x^a \xi_a,$$

i.e.

$$\xi_a = \xi_{a-1}/(1-x^a).$$

If conventionally we take $\xi_0 = 1$, we have evidently obtained Euler's generating function of linear partitions. It should be observed that the recurrence-relations (1) define ξ_a uniquely, when the initial ξ_0 is known; it is therefore sufficient in solving these relations to find a general ξ_a which satisfies (1) and reduces, when $a = 0$, to the right ξ_0.

1. Two-rowed partitions

I now consider two-rowed partitions

$$aa'a''\ldots$$
$$bb'b''\ldots$$

in which the monotonic principle is observed both vertically and horizontally. I think of such a partition as a linear partition, in doubled parts

$$\binom{a}{b}\binom{a'}{b'}\binom{a''}{b''}\ldots$$

and introduce the corresponding generating function ξ_{ab} $(a \geqslant b)$, defined by the recurrence-formula

$$\xi_{ab} = \sum x^{r+s}\xi_{rs},$$

where the summation $\sum$ is taken over all integral values of r, s subject to the inequalities

$$a \geqslant r \geqslant 0, \qquad b \geqslant s \geqslant 0, \qquad r \geqslant s.$$

Such a summation we may clearly write

$$\xi_{ab} = \sum_{s=0}^{b} \sum_{r=s}^{a} x^{r+s}\xi_{rs}. \tag{3}$$

Again it is not difficult to see that this recurrence-relation defines ξ_{ab} uniquely, once the initial ξ_{00} is given. Taking $\xi_{00} = 1$, I proceed to build up a solution.

Clearly, if the monotonic condition were waived down the columns, the two-rowed partition would be a mere combination of two linear partitions and we should have simply

$$\xi_{a^b} = \xi_a \xi_b.$$

Putting this in (3) as a trial solution, we get on the right

$$\sum_{s=0}^{b}\sum_{r=s}^{a} x^{r+s}\xi_r\xi_s = \sum_{s=0}^{b} x^s(\xi_a-\xi_{s-1})\xi_s, \text{ by (1)},$$

$$= \xi_a\xi_b - \sum_{s=1}^{b} x^s\xi_{s-1}\xi_s. \tag{4}$$

Again, taking a second trial solution

$$\xi_{ab} = \xi_{a+1}\xi_{b-1} \quad (b \geqslant 1), \qquad \xi_{a0} = 0,$$

we get on the right of (3)

$$\sum_{s=1}^{b}\sum_{r=s}^{a} x^{r+s}\xi_{r+1}\xi_{s-1} = \sum_{s=1}^{b} x^{s-1}(\xi_{a+1}-\xi_s)\xi_{s-1}$$

$$= \xi_{a+1}\xi_{b-1} - \sum_{s=1}^{b} x^{s-1}\xi_s\xi_{s-1}. \tag{5}$$

We see from (4), (5) that

$$\xi_{ab} \equiv \xi_a\xi_b - x\xi_{a+1}\xi_{b-1}, \tag{6}$$

with the convention that ξ with negative suffix vanishes, satisfies (3); and, since $\xi_{00} = 1 = \xi_0\xi_0$, it follows that (6) gives the generating function of two-rowed partitions in which the leading terms may not exceed a, b respectively.

This, as is well known, is also the generating function of two-rowed partitions in which the *numbers* of terms in the rows may not exceed a, b respectively but the *magnitudes* of the terms are unrestricted. In this form it is given by MacMahon.*

From (6) we infer that those partitions of n into two-rowed partitions with leading parts not exceeding a, b in which the monotonic condition is violated in at least one column correspond uniquely to the partitions of $n-1$ into two-rowed partitions with leading parts not exceeding $a+1$, $b-1$ in which there is no monotonic condition in the columns.

2. Many-rowed partitions

The many-rowed partition is sufficiently illustrated by consideration of the partition of three rows. The generating function ξ_{abc} of partitions in which the parts in rows 1, 2, 3 do not exceed a, b, c satisfies the recurrence-formula

$$\xi_{abc} = \sum_{t=0}^{c}\sum_{s=t}^{b}\sum_{r=s}^{a} x^{r+s+t}\xi_{rst}. \tag{7}$$

* MacMahon uses the symbol $GF(p_1\, p_2\, ;\, \infty)$ where p_1, p_2 are my a, b: loc. cit., p. 217, § 469, first formula.

I show that the solution of this recurrence-relation is

$$\xi_{rst} = \begin{vmatrix} \xi_r & \xi_{r+1} & \xi_{r+2} \\ x\xi_{s-1} & \xi_s & x^{-1}\xi_{s+1} \\ x^4\xi_{t-2} & x^2\xi_{t-1} & \xi_t \end{vmatrix}. \tag{8}$$

Substituting in the right-hand side of (7) and summing in r, we get

$$\sum_{t=0}^{c}\sum_{s=t}^{b} x^{s+t} \begin{vmatrix} \xi_a - \xi_{s-1} & x^{-1}(\xi_{a+1} - \xi_s) & x^{-2}(\xi_{a+2} - \xi_{s+1}) \\ x\xi_{s-1} & \xi_s & x^{-1}\xi_{s+1} \\ x^4\xi_{t-2} & x^2\xi_{t-1} & \xi_t \end{vmatrix}.$$

Multiply the second row of the determinant by x^{-1} and subtract from the first, and we get

$$\sum_{t=0}^{c}\sum_{s=t}^{b} x^{s+t} \begin{vmatrix} \xi_a & x^{-1}\xi_{a+1} & x^{-2}\xi_{a+2} \\ x\xi_{s-1} & \xi_s & x^{-1}\xi_{s+1} \\ x^4\xi_{t-2} & x^2\xi_{t-1} & \xi_t \end{vmatrix}.$$

Summation in s then gives

$$\sum_{t=0}^{c} x^t \begin{vmatrix} \xi_a & x^{-1}\xi_{a+1} & x^{-2}\xi_{a+2} \\ x^2(\xi_{b-1} - \xi_{t-2}) & \xi_b - \xi_{t-1} & x^{-2}(\xi_{b+1} - \xi_t) \\ x^4\xi_{t-2} & x^2\xi_{t-1} & \xi_t \end{vmatrix}$$

$$= \sum_{t=0}^{c} x^t \begin{vmatrix} \xi_a & x^{-1}\xi_{a+1} & x^{-2}\xi_{a+2} \\ x^2\xi_{b-1} & \xi_b & x^{-2}\xi_{b+1} \\ x^4\xi_{t-2} & x^2\xi_{t-1} & \xi_t \end{vmatrix}$$

$$= \begin{vmatrix} \xi_a & x^{-1}\xi_{a+1} & x^{-2}\xi_{a+2} \\ x^2\xi_{b-1} & \xi_b & x^{-2}\xi_{b+1} \\ x^6\xi_{c-2} & x^3\xi_{c-1} & \xi_c \end{vmatrix},$$

on performing the final summation. If we multiply the columns and divide the rows (in order) by 1, x, x^2 respectively, this determinant reduces to (8) with a, b, c written for r, s, t. Thus (8) satisfies the recurrence-relation (7). Moreover, it reduces to 1 when a, b, c all vanish, since $\xi_0 = 1$ and ξ with negative suffix vanishes. Hence (8) gives correctly the generating function of three-rowed partitions. The form of the determinant (8) extends to partitions in any number of rows and we have generally

$$\xi_{abcd\ldots} = \begin{vmatrix} \xi_a & \xi_{a+1} & \xi_{a+2} & \xi_{a+3}\cdots \\ x\xi_{b-1} & \xi_b & x^{-1}\xi_{b+1} & x^{-2}\xi_{b+2}\cdots \\ x^4\xi_{c-2} & x^2\xi_{c-1} & \xi_c & x^{-2}\xi_{c+1}\cdots \\ x^9\xi_{d-3} & x^6\xi_{d-2} & x^3\xi_{d-1} & \xi_d\cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}. \tag{9}$$

3. Partitions further conditioned

This formula (9) is equally applicable to other more closely conditioned partitions. Thus let us impose on the linear partition the condition that consecutive parts must differ by at least ρ, i.e. that the monotonic descent is always by steps of ρ at least.* If $\rho = 1$, for example, we have linear partitions in which repetitions are not allowed. The recurrence-formula for ξ^{ρ}_{a}, the generating function of such partitions, is evidently

$$\xi^{\rho}_{a} = \sum_{r=0}^{a} x^{r} \xi^{\rho}_{r-\rho}, \qquad (a > 0)$$

with the initial conditions:

$$\begin{aligned} \xi^{\rho}_{-r} &= 1 \qquad (0 \leqslant r \leqslant \rho) \\ &= 0 \qquad (r > \rho), \end{aligned}$$

if the final part is unrestricted; or

$$\xi^{\rho}_{-r} = 0 \qquad (r > 0),$$

if the final part must itself be ρ at least.

If $\rho = 1$, we have at once Euler's formula

$$\xi^{1}_{a} = \prod_{r=1}^{a} (1 + x^{r});$$

I do not know the general formula for $\rho > 1$.

For plane partitions we can similarly prescribe that the rows descend by steps of at least ρ and the columns by steps of at least κ. The character of the recurrence-formula is sufficiently indicated by that for the two-rowed partition:

$$\xi^{\rho,\kappa}_{ab} = \sum_{s=0}^{b} \sum_{r=s+\kappa}^{a} x^{r+s} \xi^{\rho,\kappa}_{r-\rho,s-\rho}, \qquad (10)$$

to which we must add suitable initial conditions.

It can be shown that the general recurrence-formula is satisfied by

$$\xi^{\rho,\kappa}_{abc\ldots} = \begin{vmatrix} \xi^{\rho}_{a} & \xi^{\rho}_{a-\delta} & \xi^{\rho}_{a-2\delta} \cdots \\ x^{-\delta} \xi^{\rho}_{b+\delta} & \xi^{\rho}_{b} & x^{\delta} \xi^{\rho}_{b-\delta} \cdots \\ x^{-4\delta} \xi^{\rho}_{c+2\delta} & x^{-2\delta} \xi^{\rho}_{c+\delta} & \xi^{\rho}_{c} \cdots \\ \cdot & \cdot & \cdot \end{vmatrix}, \qquad (11)$$

where $\delta \equiv \rho + \kappa - 1$. Thus the problem of determining the generating function $\xi^{\rho\kappa}_{abc\ldots}$ of the conditioned plane partition is, in general, reduced to that of determining the corresponding generating function ξ^{ρ}_{a} for

* Such a partition $aa'a''\ldots$ satisfies the inequalities

$$a \geqslant a' + \rho \geqslant a'' + 2\rho \geqslant \ldots$$

the linear partition.* For the special case $\rho+\kappa=1$, however, the formula (11) becomes indeterminate.

We may also in the many-rowed partition limit the number of columns not to exceed k. The generating function ${}_{k}\xi^{\rho,\kappa}_{ab}$ then satisfies the recurrence-formula

$$ {}_{k+1}\xi^{\rho,\kappa}_{ab} = \sum_{s=0}^{b} \sum_{r=s+\kappa}^{a} x^{r+s} \left({}_{k}\xi^{\rho,\kappa}_{r-\rho,\,s-\rho}\right). $$

The recurrence-formula for the corresponding many-rowed partition is found to be satisfied by the determinant

$$ {}_{k+1}\xi^{\rho,\kappa}_{abc\ldots} = \begin{vmatrix} {}_{k}\xi^{\rho}_{a} & {}_{k-1}\xi^{\rho}_{a-\delta} & {}_{k-2}\xi^{\rho}_{a-2\delta}\cdots \\ x^{-\delta}\left({}_{k+1}\xi^{\rho}_{b+\delta}\right) & {}_{k}\xi^{\rho}_{b} & x^{\delta}\left({}_{k-1}\xi^{\rho}_{b-\delta}\right)\cdots \\ x^{-4\delta}\left({}_{k+2}\xi^{\rho}_{c+2\delta}\right) & x^{-2\delta}\left({}_{k+1}\xi^{\rho}_{c+\delta}\right) & {}_{k}\xi^{\rho}_{c}\cdots \\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \end{vmatrix}, $$

where still $\delta \equiv \rho+\kappa-1$.

Thus again the solution of the problem for the plane partition is made to depend on the solution of the corresponding problem for the linear partition.

4. The simple plane partition

The formula (9) for the generating function of many-rowed partitions with limitation of parts in each row includes within itself MacMahon's formula for the unrestricted plane partition. We approach that formula by first taking a, b, c,... all equal. Considering, for simplicity, the three-rowed partition, write

$$ X^3_a \equiv \begin{vmatrix} \xi_a & \xi_{a+1} & \xi_{a+2} \\ x\xi_{a-1} & \xi_a & x^{-1}\xi_{a+1} \\ x^4\xi_{a-2} & x^2\xi_{a-1} & \xi_a \end{vmatrix} $$

$$ = \begin{vmatrix} (1-x^{a+1})\xi_{a+1} & \xi_{a+1} & \xi_{a+2} \\ (x-x^{a+1})\xi_a & \xi_a & x^{-1}\xi_{a+1} \\ (x^4-x^{a+3})\xi_{a-1} & x^2\xi_{a-1} & \xi_a \end{vmatrix}, \text{ since } \xi_r = (1-x^{r+1})\xi_{r+1} $$

$$ = \begin{vmatrix} \xi_{a+1} & (1-x^{a+2})\xi_{a+2} & \xi_{a+2} \\ x\xi_a & (1-x^{a+1})\xi_{a+1} & x^{-1}\xi_{a+1} \\ x^4\xi_{a-1} & (x^2-x^{a+2})\xi_a & \xi_a \end{vmatrix} $$

* We can even extend these results to 'pseudo-monotonic' partitioning, i.e. to cases ρ, κ negative, when the corresponding linear partition $aa'a''\ldots$ obeys inequalities

$$ a \geqslant a'-|\rho| \geqslant a''-2|\rho| \geqslant \ldots . $$

$$= \begin{vmatrix} (1-x^{a+2})\xi_{a+2} & \xi_{a+2} & \xi_{a+2} \\ (x-x^{a+2})\xi_{a+1} & \xi_{a+1} & x^{-1}\xi_{a+1} \\ (x^4-x^{a+4})\xi_a & x^2\xi_a & \xi_a \end{vmatrix}$$

$$= \xi_a\xi_{a+1}\xi_{a+2}\begin{vmatrix} 1 & 1 & 1 \\ x & 1 & x^{-1} \\ x^4 & x^2 & 1 \end{vmatrix}$$

$$= (1-x)^2(1-x^2)\xi_a\xi_{a+1}\xi_{a+2}.$$

The principle is clear and we thus have for the generating function of the simple k-rowed partition in which no part exceeds a

$$X_a^k = (1-x)^{k-1}(1-x^2)^{k-2}\ldots(1-x^{k-1})\xi_a\xi_{a+1}\ldots\xi_{a+k-1}$$

$$= \prod_{r=a}^{a+k-1}\xi_r \Big/ \prod_{r=0}^{k-1}\xi_r. \tag{12}$$

This is symmetrical in a, k, for we may write it in the form

$$\prod_{r=0}^{a+k-1}\xi_r \Big/ \prod_{r=0}^{a-1}\xi_r \prod_{r=0}^{k-1}\xi_r.$$

Thence we deduce

$$X_a^\infty = X_\infty^a = \prod_{r=1}^{a-1}(1-x^r)^{-r}\prod_{r=a}^{\infty}(1-x^r)^{-1} \tag{13}$$

and finally MacMahon's formula with which we started

$$X_\infty^\infty = \prod_{r=1}^{\infty}(1-x^r)^{-r}. \tag{14}$$

The formulae (12), (13), (14) are MacMahon's $GF(l; n; \infty)$, $GF(l; \infty; \infty)$ $GF(\infty; \infty; \infty)$, loc. cit., vol. ii, p. 243, § 495, the last three formulae.